Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **6**, No. 5, pp. 615-636

DOI: 10.4208/aamm.2013.m83
October 2014

A New Finite Volume Element Formulation for the Non-Stationary Navier-Stokes Equations

Zhendong Luo*

School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

Received 10 January 2013; Accepted (in revised version) 30 April 2014

Available online 23 June 2014

Abstract. A semi-discrete scheme about time for the non-stationary Navier-Stokes equations is presented firstly, then a new fully discrete finite volume element (FVE) formulation based on macroelement is directly established from the semi-discrete scheme about time. And the error estimates for the fully discrete FVE solutions are derived by means of the technique of the standard finite element method. It is shown by numerical experiments that the numerical results are consistent with theoretical conclusions. Moreover, it is shown that the FVE method is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations and it is one of the most effective numerical methods among the FVE formulation, the finite element formulation, and the finite difference scheme.

AMS subject classifications: 65N30, 65M30, 76M10

Key words: Non-stationary Navier-Stokes equations, finite volumes element method, error estimate, numerical simulations.

1 Introduction

Let $\Omega\subset\mathbb{R}^2$ be a bounded and connected polygonal domain. We consider the following incompressible non-stationary Navier-Stokes equations.

Problem 1.1. Find $\boldsymbol{u}=(u_1,u_2)$ and p such that, for $T>0$,

$$\begin{cases} \boldsymbol{u}_t-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p=\boldsymbol{f}, & (x,y,t)\in\Omega\times(0,T),\\ \nabla\cdot\boldsymbol{u}=0, & (x,y,t)\in\Omega\times(0,T),\\ \boldsymbol{u}(x,y,t)=\boldsymbol{\varphi}(x,y,t), & (x,y,t)\in\partial\Omega\times(0,T],\\ \boldsymbol{u}(x,y,0)=\boldsymbol{u}_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.1}$$

*Corresponding author.
Email: zhdluo@ncepu.edu.cn (Z. D. Luo)

where $\boldsymbol{u}=(u_1,u_2)$ represents the fluid velocity vector, p the pressure, T the total time, $\nu=1/Re$, Re the Reynolds number, $\boldsymbol{f}(x,y,t)$ the prescribed body force vector, $\boldsymbol{\varphi}(x,y,t)$ and $\boldsymbol{u}_0(x,y)$ are the boundary and initial values, respectively. For the sake of convenience, without loss of generality, we may as well suppose that $\boldsymbol{\varphi}(x,y,t)=\mathbf{0}$.

The system of the non-stationary Navier-Stokes equations is one of the important model system of equations in fluid dynamics. It has been successfully and extensively applied in many fields of practical engineering [1–3]. Due to its nonlinearity, there are no analytical solutions in general. One has to rely on numerical solutions. The finite volume element (FVE) method [4–6] is considered as one of the most effective numerical methods due to its following advantages. First, it preserves the integral invariants of conservation of mass as well as that of total energy. Second, it has higher accuracy and is more suitable for computations involving complicated boundary conditions than the finite difference (FD) method. Third, it has the same accuracy as the finite element (FE) method but is simpler and more convenient to apply than the FE method. It is also known as box method [7] or generalized difference method [8, 9]. Although it has been used to solve various types of partial differential equations, it focused on stationary partial differential equations and linear equations, for example elliptic problems, parabolic equations, Stokes equations, and viscoelastic problems, etc (see [4–17]).

Although some FVE methods for nonlinear Navier-Stokes equations have been provided, they are mainly based on stabilized and penalty FVE methods (see [18–21]). Even if the stabilized and penalty FVE methods for Navier-Stokes equations can enhance the stability of the numerical solutions and their theoretical analyses (e.g., stability and convergence) are conveniently achieved, the condition number of the coefficient matrices in their discrete systems would greatly increase. What's more, their numerical solutions would distort and diverge their accuracy solutions (in fact, the penalty term is an artificial viscosity). Thus, the theoretical study for fully discrete FVE method without any stabilization and penalty for the non-stationary Navier-Stokes equations holds more generality and more technologies required than those in [18–21]. So it has important theoretical meaning and practical value to do the theoretical analysis about the stability and error estimates of the fully discrete FVE method without any stabilization and penalty for non-stationary Navier-Stokes equations. Especially, to the best of our knowledge, as so far, there are no relative results published to do directly the theoretical study for the fully discrete FVE method without any stabilization and penalty for non-stationary Navier-Stokes equations. Therefore, we will do these studies in this paper and provide the numerical experiments for illustrating the feasibility and efficiency of FVE method without any stabilization and penalty. It is also shown that the FVE method is more stabile than FE method and FD scheme by comparing their numerical solutions. Especially, we here directly derive a new fully discrete FVE formulation based on macroelement without any stabilization and penalty from the semi-discrete formulation with respect to time and do theoretical study which could avoid the semi-discrete FVE formulation about spatial variable and satisfy discrete Babuška-Brezzi (B-B) inequality (see [23, 24]).

It is a new study attempt for the non-stationary Navier-Stokes equations.

The plan of this paper is organized as follows. In Section 2, a semi-discrete formulation with respect to time for the non-stationary Navier-Stokes equations and its error estimates are provided. In Section 3, a new fully discrete FVE formulation based on macroelement without any stabilization and penalty for the non-stationary Navier-Stokes equation is directly established from the semi-discrete scheme about time. It avoids the semi-discrete FVE formulation about spatial variable and satisfies discrete B-B inequality. In Section 4, the theoretical analysis about the existence and uniqueness and error estimates for the fully discrete FVE solutions are derived by means of the standard FE method. In Section 5, some numerical experiments are provided for illustrating that the numerical errors between the fully discrete FVE solutions and the the accuracy solutions are consistent with the theoretical results obtained previously, that FVE method is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations, and that the FVE formulation is one of the most effective numerical methods by comparing the FVE solutions with the FE solutions and the FD solutions for the non-stationary Navier-Stokes equations. Section 6 provides main conclusions and discussion.

2 Semi-discrete formulation about time and error estimate for the non-stationary Navier-Stokes equations

The Sobolev spaces in this paper are standard (see [22]). Let $U=H_0^1(\Omega)^2$, $M=L_0^2(\Omega)=\{q\in L^2(\Omega);\ \int_\Omega q dxdy=0\}$. Then the variational formulation for Problem 1.1 is as follows.

Problem 2.1. Find $(\boldsymbol{u}(t),p(t)):[0,T]\to U\times M$ such that

$$\begin{cases} (\boldsymbol{u}_t,\boldsymbol{v})+a(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{v},p)=(\boldsymbol{f},\boldsymbol{v}), & \forall \boldsymbol{v}\in U,\\ b(\boldsymbol{u},q)=0, & \forall q\in M,\\ \boldsymbol{u}(x,y,0)=\boldsymbol{u}_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{2.1}$$

where $(\cdot,\cdot)$ denotes L^2 inner product, $a(\boldsymbol{u},\boldsymbol{v})=\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})$, $b(\boldsymbol{v},q)=(\mathrm{div}\boldsymbol{v},q)$, and $a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})+((\mathrm{div}\boldsymbol{u})\boldsymbol{v},\boldsymbol{w})/2=[(\boldsymbol{u}\nabla\boldsymbol{v},\boldsymbol{w})-(\boldsymbol{u}\nabla\boldsymbol{w},\boldsymbol{v})]/2$.

The following property for trilinear form $a_1(\cdot,\cdot,\cdot)$ is often used (see [1–3,13–16,23]).

$$a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-a_1(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0, \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in U, \tag{2.2a}$$

$$\begin{aligned}&|a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|+|a_1(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})|\\ \le{}& \tilde{C}_1\big(\|\boldsymbol{u}\|_0^{\frac12}|\boldsymbol{u}|_1^{\frac12}|\boldsymbol{v}|_1+|\boldsymbol{u}|_1\|\boldsymbol{v}\|_0^{\frac12}|\boldsymbol{v}|_1^{\frac12}\big)\|\boldsymbol{w}\|_0^{\frac12}|\boldsymbol{w}|_1^{\frac12}, \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in U,\end{aligned} \tag{2.2b}$$

$$\begin{aligned}&|a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|+|a_1(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})|+|a_1(\boldsymbol{v},\boldsymbol{w},\boldsymbol{u})|\\ \le{}& \tilde{C}_2(|\boldsymbol{u}|_1\|\boldsymbol{v}\|_0^{\frac12}|\boldsymbol{v}|_1^{\frac12}+\|\boldsymbol{u}\|_0^{\frac12}|\boldsymbol{u}|_1^{\frac12}|\boldsymbol{v}|_1)\|\boldsymbol{w}\|_0, \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in U,\end{aligned} \tag{2.2c}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are two constants independent of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

The bilinear form $a(\cdot,\cdot)$ has the following properties (see [1–3, 13–16, 23]):

$$a(\boldsymbol{v},\boldsymbol{v})=\nu|\boldsymbol{v}|_1^2, \quad \forall \boldsymbol{v}\in U; \quad |a(\boldsymbol{u},\boldsymbol{v})|\le \nu|\boldsymbol{u}|_1|\boldsymbol{v}|_1, \quad \forall \boldsymbol{u},\, \boldsymbol{v}\in U. \tag{2.3}$$

The bilinear form $b(\cdot,\cdot)$ satisfies the following continuous B-B inequality (see [1–3, 13–16, 23])

$$\sup_{\boldsymbol{v}\in U}\frac{b(q,\boldsymbol{v})}{|\boldsymbol{v}|_1}\ge \beta\|q\|_0, \quad \forall q\in M, \tag{2.4}$$

where β is a constant independent of $\boldsymbol{v}$ and q.

Define

$$N_0=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in U}\frac{a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})}{|\boldsymbol{u}|_1\cdot|\boldsymbol{v}|_1\cdot|\boldsymbol{w}|_1}. \tag{2.5}$$

Thanks to (2.2a)-(2.5), when $N_0\nu^{-1}\|\boldsymbol{f}\|_{-1}\le 1$, Problem 2.1 has a unique solution, and there holds the following result (see [1–3, 13–16, 23]):

$$\begin{aligned}&\|\boldsymbol{u}\|_0+\|\boldsymbol{u}_t\|_{L^2(L^2)}+\|\nabla\boldsymbol{u}\|_{L^2(L^2)}+\|p\|_{L^2(L^2)}\\ \le& C\big(\|\boldsymbol{u}_0\|_1+\|\boldsymbol{f}\|_{L^2(L^2)}+N_0\|\boldsymbol{f}\|^2_{L^2(L^2)}+\|\boldsymbol{f}\|_{L^2(H^{-1})}\big),\end{aligned} \tag{2.6}$$

where $\|\cdot\|_{H^m(H^l)}$ is the norm of $H^m(0,T;H^l(\Omega))$ or $H^m(0,T;H^l(\Omega)^2)^2$ ($m\ge 0$ and $l\ge -1$) and C is a constant.

For given positive integer N, let $k=T/N$ denote time step, $\boldsymbol{u}^n$ be the semi-discrete approximation of $\boldsymbol{u}$ at $t_n\equiv nk$ ($n=0,1,\cdots,N$). Let $\bar{\partial}_t\boldsymbol{u}^n=(\boldsymbol{u}^n-\boldsymbol{u}^{n-1})/k$ denote the approximation of $\boldsymbol{u}_t$, then the semi-discrete approximation scheme about time reads as follows.

Problem 2.2. Find $(\boldsymbol{u}^n,p^n)\in U\times M$ ($1\le n\le N$) such that

$$\begin{cases}(\bar{\partial}_t\boldsymbol{u}^n,\boldsymbol{v})+a(\boldsymbol{u}^n,\boldsymbol{v})+a_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v})-b(\boldsymbol{v},p^n)=(\boldsymbol{f}^n,\boldsymbol{v}), & \forall \boldsymbol{v}\in U,\\ b(\boldsymbol{u}^n,q)=0, & \forall q\in M,\\ \boldsymbol{u}^0=\boldsymbol{u}_0(x,y), & (x,y)\in\Omega,\end{cases} \tag{2.7}$$

where $\boldsymbol{f}^n=\boldsymbol{f}(t_n,x,y)$ is the value of $\boldsymbol{f}(t,x,y)$ at point t_n.

There holds the following theorem for the semi-discrete formulation about time namely Problem 2.2.

Theorem 2.1. *If $\boldsymbol{u}_0\in H^2(\Omega)^2$ and $\boldsymbol{f}\in L^2(0,T;H^{-1}(\Omega)^2)^2$, then Problem 2.2 has a unique solution series $(\boldsymbol{u}^n,p^n)\in U\times M$ ($n=1,2,\cdots,N$) which satisfied the following stability*

$$\|\boldsymbol{u}^n\|_0^2+\nu k\sum_{i=1}^n\|\nabla\boldsymbol{u}^i\|_0^2\le\|\boldsymbol{u}_0\|_0^2+\nu^{-1}k\sum_{i=1}^n\|\boldsymbol{f}^i\|_{-1}^2, \tag{2.8a}$$

$$\begin{aligned}k\sum_{i=1}^n\|p^i\|_0\le&\beta^{-1}\Big(2\|\boldsymbol{u}_0\|_0+kN_0\|\nabla\boldsymbol{u}_0\|_0^2+2N_0\nu^{-1}\|\boldsymbol{u}_0\|_0^2\Big)\\ &+k\beta^{-1}N_0\nu^{-2}\sum_{i=1}^n\|\boldsymbol{f}^i\|_{-1}^2+\beta^{-1}\Big(1+2\sqrt{kn\nu}\Big)\Big(k\nu^{-1}\sum_{i=1}^n\|\boldsymbol{f}^i\|_{-1}^2\Big)^{1/2}.\end{aligned} \tag{2.8b}$$

And when $\boldsymbol{f}\in W^{2,\infty}(0,T;H^1(\Omega)^2)^2$ *and* $N_0\nu^{-1}\|\nabla\boldsymbol{u}(t)\|_0\le 1/2$, *there hold the following error estimates:*

$$\|\boldsymbol{u}(t_n)-\boldsymbol{u}^n\|_1\le C_0 k,\quad \|p(t_n)-p^n\|_0\le\tilde{C}k,\quad n=1,2,\cdots N, \tag{2.9}$$

where

$$\begin{aligned}
&C_0=2\nu\big[\|\boldsymbol{u}(t)\|_{W^{2,\infty}(H^{-1})}+N_0\|\nabla\boldsymbol{u}(t_n)\|_0\|\nabla\boldsymbol{u}_t\|_{L^\infty(L^2)}\big],\\
&\tilde{C}=\big[\|\boldsymbol{u}(t)\|_{W^{2,\infty}(H^{-1})}+N_0\|\nabla\boldsymbol{u}(t)\|_{W^{1,\infty}(L^2)}\|\nabla\boldsymbol{u}(t_n)\|_0+C_0(\nu+N_0\|\nabla\boldsymbol{u}(t_n)\|_0)\big]/\beta,
\end{aligned}$$

are two constants independent of k.

Proof. Let $A(\boldsymbol{u}^n,\boldsymbol{v})=(\boldsymbol{u}^n,\boldsymbol{v})+ka(\boldsymbol{u}^n,\boldsymbol{v})+ka_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v})$, $F(\boldsymbol{v})=(k\boldsymbol{f}^n+\boldsymbol{u}^{n-1},\boldsymbol{v})$. Then for given $\boldsymbol{u}^{n-1}$ and fixed n and k, $A(\boldsymbol{u}^n,\boldsymbol{v})$ is bounded bilinear form and satisfies

$$A(\boldsymbol{u}^n,\boldsymbol{u}^n)=\|\boldsymbol{u}^n\|_0^2+k\nu\|\nabla\boldsymbol{u}^n\|_0^2\ge\alpha_0\|\boldsymbol{u}^n\|_1^2, \tag{2.10}$$

where $\alpha_0=\min\{k\nu,1\}$ is a constant independent of $\boldsymbol{u}^n$. For given $\boldsymbol{u}^{n-1}$ and $\boldsymbol{f}$, $F(\boldsymbol{v})$ is a continuous function on U, we know from (2.4) that $kb(\boldsymbol{v},q)$ satisfies B-B condition too. Then Problem 2.2 has a unique series of solutions $(\boldsymbol{u}^n,p^n)\in U\times M$ $(n=1,2,\cdots,N)$ from the theory of existence and uniqueness of solution for mixed variational problem (see [23–25]).

Let $\boldsymbol{v}=\boldsymbol{u}^n$ in Problem 2.2. With (2.2a) and Hölder and Cauchy inequalities, we have

$$\begin{aligned}
\|\boldsymbol{u}^n\|_0^2+k\nu\|\nabla\boldsymbol{u}^n\|_0^2&=(\boldsymbol{u}^n,\boldsymbol{u}^n)+ka(\boldsymbol{u}^n,\boldsymbol{u}^n)=k(\boldsymbol{f}^n,\boldsymbol{u}^n)+(\boldsymbol{u}^{n-1},\boldsymbol{u}^n)\\
&\le k\|\boldsymbol{f}^n\|_{-1}\|\nabla\boldsymbol{u}^n\|_0+\|\boldsymbol{u}^{n-1}\|_0\|\boldsymbol{u}^n\|_0\\
&\le\frac{1}{2}\big[k\nu^{-1}\|\boldsymbol{f}^n\|_{-1}^2+k\nu\|\nabla\boldsymbol{u}^n\|_0^2+\|\boldsymbol{u}^{n-1}\|_0^2+\|\boldsymbol{u}^n\|_0^2\big].
\end{aligned} \tag{2.11}$$

From (2.11), we obtain that

$$\|\boldsymbol{u}^n\|_0^2+k\nu\|\nabla\boldsymbol{u}^n\|_0^2\le k\nu^{-1}\|\boldsymbol{f}^n\|_{-1}^2+\|\boldsymbol{u}^{n-1}\|_0^2. \tag{2.12}$$

Summing (2.12) from 1 to n yields (2.8a).

With (2.4) and Problem 2.2, we have

$$\begin{aligned}
k\beta\|p^n\|_0&\le\sup_{\boldsymbol{v}\in U}\frac{kb(\boldsymbol{v},p^n)}{\|\nabla\boldsymbol{v}\|_0}\\
&=\sup_{\boldsymbol{v}\in U}\frac{(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{v})+ka(\boldsymbol{u}^n,\boldsymbol{v})+ka_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v})-k(\boldsymbol{f}^n,\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_0}.
\end{aligned} \tag{2.13}$$

By summing (2.19) from 1 to n and using with Hölder inequality, we obtain

$$\begin{aligned}
k\beta\sum_{i=1}^{n}\|p^i\|_0 \le& \|\boldsymbol{u}^n\|_{-1}+\|\boldsymbol{u}_0\|_{-1}+k\nu\sum_{i=1}^{n}\|\nabla\boldsymbol{u}^i\|_0+kN_0\sum_{i=1}^{n}\|\nabla\boldsymbol{u}^{i-1}\|_0\|\nabla\boldsymbol{u}^i\|_0+k\sum_{i=1}^{n}\|\boldsymbol{f}^i\|_{-1}\\
\le&\|\boldsymbol{u}^n\|_0+\|\boldsymbol{u}_0\|_0+kN_0\|\nabla\boldsymbol{u}_0\|_0^2+k\sum_{i=1}^{n}\|\boldsymbol{f}^i\|_{-1}+2kN_0\sum_{i=1}^{n}\|\nabla\boldsymbol{u}^i\|_0^2\\
&+\sqrt{kn\nu}\Big(k\nu\sum_{i=1}^{n}\|\nabla\boldsymbol{u}^i\|_0^2\Big)^{1/2}\\
\le&2\|\boldsymbol{u}_0\|_0+kN_0\|\nabla\boldsymbol{u}_0\|_0^2+2N_0\nu^{-1}\|\boldsymbol{u}_0\|_0^2+kN_0\nu^{-2}\sum_{i=1}^{n}\|\boldsymbol{f}^i\|_{-1}^2\\
&+\big(1+2\sqrt{kn\nu}\big)\Big(k\nu^{-1}\sum_{i=1}^{n}\|\boldsymbol{f}^i\|_{-1}^2\Big)^{1/2}. \qquad (2.14)
\end{aligned}$$

Combining (2.14) and (2.8a) yields (2.8b).

Subtracting Problem 2.2 from Problem 2.1 taking $t=t_n$ and then taking $\boldsymbol{v}=\boldsymbol{u}(t_n)-\boldsymbol{u}^n$ and $q=p(t_n)-p^n$, using Taylor's formula, we obtain

$$\begin{aligned}
&\nu\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0^2=a(\boldsymbol{u}(t_n)-\boldsymbol{u}^n,\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\\
=&(\boldsymbol{R}(t),\boldsymbol{u}(t_n)-\boldsymbol{u}^n)-a_1(\boldsymbol{u}(t_n)-\boldsymbol{u}^{n-1},\boldsymbol{u}(t_n),\boldsymbol{u}(t_n)-\boldsymbol{u}^n),\quad t\in[t_{n-1},t_n], \qquad (2.15)
\end{aligned}$$

where $\boldsymbol{R}(t)=\int_{t_{n-1}}^{t}\boldsymbol{u}^{(2)}(s)(t-s)ds/k$. If $\boldsymbol{f}\in W^{1,\infty}(0,T;H^1(\Omega)^2)^2$, we have $\boldsymbol{u}\in W^{2,\infty}(0,T;H^2(\Omega)^2\cap H_0^1(\Omega)^2)^2$ by the regularity (see [1–3,23]). Then from Hölder inequality, we get that

$$\begin{aligned}
\nu\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0^2=&(R(t),\boldsymbol{u}(t_n)-\boldsymbol{u}^n)-a_1(\boldsymbol{u}(t_n)-\boldsymbol{u}^n,\boldsymbol{u}(t_n),\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\\
&-a_1(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{u}(t_n),\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\\
\le&k\|\boldsymbol{u}(t)\|_{W^{2,\infty}(H^{-1})}\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0+N_0\|\nabla\boldsymbol{u}(t_n)\|_0\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0^2\\
&+kN_0\|\nabla\boldsymbol{u}(t_n)\|_0\|\nabla\boldsymbol{u}_t\|_{L^\infty(L^2)}\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0. \qquad (2.16)
\end{aligned}$$

And if $N_0\nu^{-1}\|\nabla\boldsymbol{u}(t)\|_0\le 1/2$, we have

$$\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0\le C_0k,\quad n=0,1,2,\cdots,N, \qquad (2.17)$$

where $C_0=2\nu[\|\boldsymbol{u}(t)\|_{W^{2,\infty}(H^{-1})}+N_0\|\nabla\boldsymbol{u}(t_n)\|_0\|\nabla\boldsymbol{u}_t\|_{L^\infty(L^2)}]$ is a constant independent of k. Subtracting Problem 2.2 from Problem 2.1 taking $t=t_n$, and using Taylor's formula yield that

$$\begin{aligned}
&(\boldsymbol{R},\boldsymbol{v})+a(\boldsymbol{u}(t_n)-\boldsymbol{u}^n,\boldsymbol{v})+a_1(\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n-1}),\boldsymbol{u}(t_n),\boldsymbol{v})+a_1(\boldsymbol{u}(t_{n-1})-\boldsymbol{u}^{n-1},\boldsymbol{u}(t_n),\boldsymbol{v})\\
=&b(\boldsymbol{v},p(t_n)-p^n),\quad t\in[t_{n-1},t_n]. \qquad (2.18)
\end{aligned}$$

Then, if $\boldsymbol{f}\in W^{1,\infty}(0,T;H^1(\Omega)^2)^2$, $\boldsymbol{u}\in W^{2,\infty}(0,T;H^2(\Omega)^2\cap H_0^1(\Omega)^2)^2$, with (2.4), Hölder inequality, and (2.17), we have

$$
\begin{aligned}
&\|p(t_n)-p^n\|_0\le\beta^{-1}\sup_{v\in U}\frac{b(\boldsymbol{v},p(t_n)-p^n)}{\|\nabla v\|_0}\\
=&\beta^{-1}\sup_{v\in U}[(\boldsymbol{R},\boldsymbol{v})+a(\boldsymbol{u}(t_n)-\boldsymbol{u}^n,\boldsymbol{v})+a_1(\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n-1}),\boldsymbol{u}(t_n),\boldsymbol{v})\\
&+a_1(\boldsymbol{u}(t_{n-1})-\boldsymbol{u}^{n-1},\boldsymbol{u}(t_n),\boldsymbol{v})]/\|\nabla v\|_0\\
\le&\beta^{-1}\big[k\|\boldsymbol{u}(t)\|_{W^{2,\infty}(H^{-1})}+N_0k\|\nabla\boldsymbol{u}(t)\|_{W^{1,\infty}(L^2)}\|\nabla\boldsymbol{u}(t_n)\|_0\\
&+\nu\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0+N_0\|\nabla(\boldsymbol{u}(t_{n-1})-\boldsymbol{u}^{n-1})\|_0\|\nabla\boldsymbol{u}(t_n)\|_0\big]\\
\le&\tilde{C}k,
\end{aligned}\tag{2.19}
$$

where $\tilde{C}=[\|\boldsymbol{u}(t)\|_{W^{2,\infty}(H^{-1})}+N_0\|\nabla\boldsymbol{u}(t)\|_{W^{1,\infty}(L^2)}\|\nabla\boldsymbol{u}(t_n)\|_0+C_0(N_0\|\nabla\boldsymbol{u}(t_n)\|_0+\nu)]/\beta$ is a constant independent of k. The proof of Theorem 2.1 is finished. $\square$

3 A new fully discrete FVE formulation for the non-stationary Navier-Stokes equations

3.1 The basic theory of FVE method

In order to construct the fully discrete FVE formulation for the non-stationary Navier-Stokes equations, it is necessary to introduce the triangulation and dual partition for computational field $\overline{\Omega}$ (more details see [6–9, 23–25]).

Firstly, let $\Im_{2h}=\{\tilde{K}\}$ be a quasi-uniform triangulation of $\overline{\Omega}$ with maximum diameter $2h=\max\{2h_{\tilde{K}}\}$, where $2h_{\tilde{K}}$ is the diameter of the triangle $\tilde{K}\in\Im_{2h}$, and the interior angle of any triangle $\tilde{K}\in\Im_{2h}$ is smaller than $\pi/2$. For any $\tilde{K}\in\Im_{2h}$, connect its third midpoints cutting $\tilde{K}$ into four triangulations which reconstructs a quasi-uniform triangulation $\Im_h=\{K\}$ of $\overline{\Omega}$ with maximum diameter $h=\max h_K$, where h_K is the diameter of triangle $K\in\Im_h$. Let $Z_h\equiv\{\boldsymbol{z}_i\}_{i=1}^{M_1+M_2}$ ($\boldsymbol{z}_i=(x_i,y_i)$) be the nodal points of triangulation $\Im_h$, where $Z_h^\circ\equiv\{\boldsymbol{z}_i\}_{i=1}^{M_1}$ is the interior nodal points of the triangulation $\Im_h$, and $\{\boldsymbol{z}_i\}_{i=M_1+1}^{M_2}$ the nodal points on $\partial\Omega$.

Next, we introduce a dual partition $\Im_h^*$ based on $\Im_h$, whose element are called the control volumes. We construct the control volume in the same way as in [6–9]. Let $\boldsymbol{z}_K=(x_K,y_k)$ be the barycenter of $K\in\Im_h$. We connect $\boldsymbol{z}_K$ with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z ($\boldsymbol{z}=(x,y)\in Z_h(K)$, where $Z_h(K)$ are the vertices of K). Then with each vertex $\boldsymbol{z}\in Z_h=\bigcup_{K\in\Im_h}Z_h(K)$, we associate a control volume V_z, which consists of the union of the sub-regions K_z, sharing the vertex $\boldsymbol{z}=(x_z,y_z)$. Finally, we obtain a group of control volumes covering the domain $\overline{\Omega}$, which is called a barycenter-type dual partition $\Im_h^*$ of the triangulation $\Im_h$ (see Fig. 1).

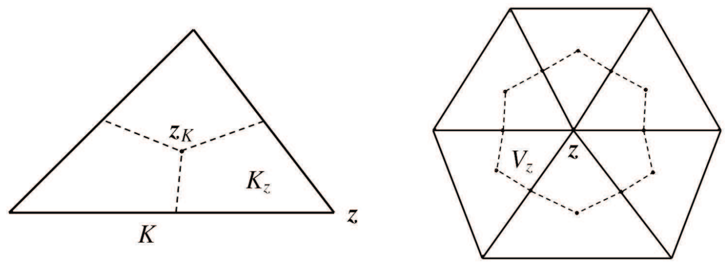

Figure 1: Left chart is a triangle K partitioned into three sub-regions K_z. Right chart is a sample region with dotted lines indication the corresponding control volume V_z.

The dual partition $\Im_h^*$ is known as quasi-uniform, if there exist two positive constants $\bar{C}_1$ and $\bar{C}_2$ independent of spatial mesh size h such that

$$\bar{C}_1 h^2 \le \mathrm{mes}(V_z) \le \bar{C}_2 h^2, \quad \forall V_z \in \Im_h^*,$$

where $\mathrm{mes}(V_z)$ denotes the measure of V_z. If the triangulation $\Im_h$ is quasi-uniform, then the dual partition $\Im_h^*$ is also quasi-uniform (see [6–9, 23–25]).

The trial function spaces U_h of velocity and M_h of pressure are respectively defined as follows:

$$\begin{aligned} U_h &= \{\boldsymbol{v}_h \in X \cap C(\overline{\Omega})^2;\ \boldsymbol{v}_h|_K \in \mathcal{P}_1^2(K),\ \forall K \in \Im_h\}, \\ M_h &= \{q_h \in L_0^2(\Omega);\ q_h|_K \in \mathcal{P}_0(K),\ \forall K \in \Im_{2h}\}, \end{aligned}$$

where $\mathcal{P}_l$ $(l=0,1)$ are l-th polynomial spaces on K. It is obvious that $U_h \subset U = H_0^1(\Omega)^2$ and U_h and M_h satisfy the following discrete B-B inequality (see [23, 24, 26])

$$\sup_{\boldsymbol{v}_h \in U_h} \frac{|b(q_h, \boldsymbol{v}_h)|}{|\boldsymbol{v}_h|_1} \ge \tilde{\beta} \|q_h\|_0, \quad \forall q_h \in M_h, \tag{3.1}$$

where $\tilde{\beta}$ is a constant independent of h and k.

Let Π_h be the interpolation projection of U onto U_h, then if $\boldsymbol{u} \in H^2(\Omega)$, it follows from the interpolation theory on finite element spaces (see [6–9, 23–25]) that

$$|\boldsymbol{u} - \Pi_h \boldsymbol{u}|_m \le C h^{2-m} |\boldsymbol{u}|_2, \quad m = 0, 1, \tag{3.2}$$

where C in this context indicates a positive constant which is possibly different at different occurrence, being independent of spatial mesh size h and time step k. The test function space $\tilde{U}_h$ is chosen as follows

$$\tilde{U}_h = \{\boldsymbol{v}_h \in L^2(\Omega)^2;\ \boldsymbol{v}_h|_{V_z} \in P_0(V_z)^2 (\forall V_z \in \Im_h^*),\ \boldsymbol{v}_h|_{V_z} = \boldsymbol{0}(V_z \cap \partial\Omega \ne \phi)\}, \tag{3.3}$$

which is spanned by the following functions

$$\phi_z(x,y) = \begin{cases} 1, & (x,y) \in V_z, \\ 0, & (x,y) \notin V_z, \end{cases} \quad \forall \boldsymbol{z} = (x_z, y_z) \in Z_h^\circ. \tag{3.4}$$

For $\boldsymbol{w}\in U$, let $\Pi_h^*\boldsymbol{w}$ is interpolation projector of $\boldsymbol{w}$ on $\tilde{U}_h$, i.e.,

$$\Pi_h^*\boldsymbol{w}=\sum_{\boldsymbol{z}\in Z_h^\circ}\boldsymbol{w}(\boldsymbol{z})\phi_z. \tag{3.5}$$

By the interpolation theory of Sobolev space [9, 23–25], we have

$$\|\boldsymbol{w}-\Pi_h^*\boldsymbol{w}\|_0\le Ch|\boldsymbol{w}|_1. \tag{3.6}$$

Moveover, the interpolation projection Π_h^* satisfies the following property (see [11]).

Lemma 3.1. *If $\boldsymbol{v}_h\in U_h$, then*

$$\int_K(\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h)dxdy=0,\quad K\in\Im_h;\quad \|\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h\|_{L^r(\Omega)}\le Ch_K\|\boldsymbol{v}_h\|_{W^{1,r}(\Omega)},\quad 1\le r\le\infty.$$

From [9, 11, 18–21, 23], we have the following two lemmas.

Lemma 3.2. *For $K\in\Im_h$ and $\boldsymbol{z}\in Z_h$, let $S_z^*=\mathrm{mes}(V_z)$, $S_K=\mathrm{mes}(K)$, and $\boldsymbol{z}_i$, $\boldsymbol{z}_j$, and $\boldsymbol{z}_k$ are the three vertices of K,*

$$\|\boldsymbol{u}_h\|_{0,h}^2\equiv\|\Pi_h^*\boldsymbol{u}_h\|_0^2=\sum_{V_z\in\Im_h^*}\boldsymbol{u}_h^2(z)S_z^*=\frac{1}{3}\sum_{K\in\Im_h}[\boldsymbol{u}_h^2(\boldsymbol{z}_i)+\boldsymbol{u}_h^2(\boldsymbol{z}_j)+\boldsymbol{u}_h^2(\boldsymbol{z}_k)]S_K, \tag{3.7a}$$

$$|\boldsymbol{u}_h|_{1,h}^2\equiv\sum_{z\in K\in\Im_h}\Big[\Big(\frac{\partial\boldsymbol{u}_h(\boldsymbol{z})}{\partial x}\Big)^2+\Big(\frac{\partial\boldsymbol{u}_h(\boldsymbol{z})}{\partial y}\Big)^2\Big]S_K, \tag{3.7b}$$

$$\|\boldsymbol{u}_h\|_{1,h}^2=\|\boldsymbol{u}_h\|_{0,h}^2+|\boldsymbol{u}_h|_{1,h}^2. \tag{3.7c}$$

Then the pairs of norms $|\cdot|_{1,h}$ and $|\cdot|_1$, $\|\cdot\|_{0,h}$ and $|\cdot|_0$, $\|\cdot\|_{1,h}$ and $\|\cdot\|_1$ on U_h are equivalent, respectively.

Lemma 3.3. *There holds the following statement*

$$(\boldsymbol{u}_h,\Pi_h^*\boldsymbol{v}_h)=(\boldsymbol{v}_h,\Pi_h^*\boldsymbol{u}_h),\quad\forall\boldsymbol{u}_h,\boldsymbol{v}_h\in U_h. \tag{3.8}$$

For any $\boldsymbol{u}\in H^m(\Omega)^2$ $(m=0,1)$ and $\boldsymbol{v}_h\in U_h$,

$$|(\boldsymbol{u},\boldsymbol{v}_h)-(\boldsymbol{u},\Pi_h^*\boldsymbol{v}_h)|\le Ch^{m+n}\|\boldsymbol{u}\|_m\|\boldsymbol{v}_h\|_n,\quad n=0,1. \tag{3.9}$$

Set $|||\boldsymbol{u}_h|||_0=(\boldsymbol{u}_h,\Pi_h^*\boldsymbol{u}_h)^{1/2}$, then $|||\cdot|||_0$ is equivalent to $\|\cdot\|_0$ on U_h, i.e., there exist two positive constants C_3 and C_4 independent of mesh size h and k, such that

$$C_3\|\boldsymbol{u}_h\|_0\le|||\boldsymbol{u}_h|||_0\le C_4\|\boldsymbol{u}_h\|_0,\quad\forall\boldsymbol{u}_h\in U_h. \tag{3.10}$$

3.2 A new fully discrete FVE formulation for the non-stationary Navier-Stokes equations

Though the trial function space U_h satisfies $U_h\subset U$ (conforming element) like finite element method, the test function space $\tilde{U}_h\not\subset U_h$. As in the case of nonconforming FE methods, this is due to the loss of continuity of the vector functions in $\tilde{U}_h$ on the boundary of two neighboring elements. So the bilinear forms $a(\boldsymbol{u},\boldsymbol{v})$ and $b(\boldsymbol{v},p)$ in Problem 2.1 must be revised accordingly. By using Green's formulation, we have that

$$\int_{V_z}\Delta\boldsymbol{u}\cdot\boldsymbol{v}dxdy=-\int_{V_z}\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}dxdy+\int_{\partial V_z}(\boldsymbol{v}\nabla\boldsymbol{u})\cdot\boldsymbol{n}ds, \tag{3.11a}$$

$$\int_{V_z}\nabla p\cdot\boldsymbol{v}dxdy=-\int_{V_z}p\mathrm{div}\boldsymbol{v}dxdy+\int_{\partial V_z}p\boldsymbol{v}\cdot\boldsymbol{n}ds, \tag{3.11b}$$

where $\int_{\partial V_z}$ denotes the line integrals, with the counter clockwise direction on the boundary ∂V_z of dual element; $\boldsymbol{n}=(n_1,n_2)^T$ denotes the unit outer normal vector to ∂V_z. So the bilinear forms $a(\boldsymbol{u},\boldsymbol{v})$ and $b(\boldsymbol{v},p)$ in Problem 2.1 can be rewritten as

$$a(\boldsymbol{u},\boldsymbol{v})=\nu\sum_{V_z\in\Im_h^*}\Big[\int_{V_z}\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}dxdy-\int_{\partial V_z}(\boldsymbol{v}\nabla\boldsymbol{u})\cdot\boldsymbol{n}ds\Big], \tag{3.12a}$$

$$b(\boldsymbol{v},p)=-\sum_{V_z\in\Im_h^*}\Big[\int_{\partial V_z}p\boldsymbol{v}\cdot\boldsymbol{n}ds-\int_{V_z}p\mathrm{div}\boldsymbol{v}dxdy\Big]. \tag{3.12b}$$

Since $\tilde{U}_h$ is the piecewise constant vector function space with the characteristic functions of the dual elements V_z as the basis functions, we have that

$$a(\boldsymbol{u},\boldsymbol{v})=-\nu\sum_{V_z\in\Im_h^*}\int_{\partial V_z}(\boldsymbol{v}\nabla\boldsymbol{u})\cdot\boldsymbol{n}ds,\qquad\forall\boldsymbol{u}\in U,\quad\forall\boldsymbol{v}\in\tilde{U}_h, \tag{3.13a}$$

$$b(\boldsymbol{v},p)=-\sum_{V_z\in\Im_h^*}\int_{\partial V_z}p\boldsymbol{v}\cdot\boldsymbol{n}ds,\qquad\forall p\in M,\quad\forall\boldsymbol{v}\in\tilde{U}_h. \tag{3.13b}$$

Put

$$a_h(\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)=-\nu\sum_{j=1}^{M_1}\int_{\partial V_{z_j}}(\boldsymbol{v}_h(\boldsymbol{z}_j)\nabla\boldsymbol{u}_h^n)\cdot\boldsymbol{n}ds, \tag{3.14a}$$

$$b_h(\Pi_h^*\boldsymbol{v}_h,q_h)=-\sum_{j=1}^{M_1}\boldsymbol{v}_h(\boldsymbol{z}_j)\int_{\partial V_{z_j}}q_h\boldsymbol{n}ds. \tag{3.14b}$$

Thus, from [9, 11, 18–21, 23], there holds the following lemma.

Lemma 3.4. *There hold the following results:*

$$a_h(\boldsymbol{u}_h,\Pi_h^*\boldsymbol{v}_h)=a(\boldsymbol{u}_h,\boldsymbol{v}_h),\qquad\forall\boldsymbol{u}_h,\boldsymbol{v}_h\in U_h, \tag{3.15a}$$

$$b_h(\Pi_h^*\boldsymbol{v}_h,p_h)=-b(\boldsymbol{v}_h,p_h),\qquad\forall\boldsymbol{v}_h\in U_h,\quad\forall p_h\in M_h. \tag{3.15b}$$

Further, $a_h(\boldsymbol{u}_h,\Pi_h^*v_h)$ is symmetric, bounded, and positive definite, i.e.,

$$a_h(\boldsymbol{u}_h,\Pi_h^*\boldsymbol{v}_h)=a_h(\boldsymbol{v}_h,\Pi_h^*\boldsymbol{u}_h),\quad \forall \boldsymbol{u}_h,\boldsymbol{v}_h\in U_h, \tag{3.16}$$

and there exist three positive constants h_0, C_0, and $\tilde{C}_0$ such that, when $0<h\le h_0$,

$$a_h(\boldsymbol{u}_h,\Pi_h^*\boldsymbol{u}_h)\ge\nu|\boldsymbol{u}_h|_1^2,\quad |a_h(\boldsymbol{u}_h,\Pi_h^*\boldsymbol{v}_h)|\le C_0\|\boldsymbol{u}_h\|_1\|\boldsymbol{v}_h\|_1,\quad \forall \boldsymbol{u}_h,\boldsymbol{v}_h\in U_h. \tag{3.17}$$

There exists a constant $\tilde{\beta}>0$ independent of h and k such that

$$\sup_{\boldsymbol{v}_h\in U_h}\frac{|b_h(\Pi_h^*\boldsymbol{v}_h,p_h)|}{\|\boldsymbol{v}_h\|_1}=\sup_{\boldsymbol{v}_h\in U_h}\frac{|b(\boldsymbol{v}_h,p_h)|}{\|\boldsymbol{v}_h\|_1}\ge\tilde{\beta}\|p_h\|_0,\quad \forall p_h\in M_h. \tag{3.18}$$

Then a new fully discrete FVE formulation based on macroelement for Problem 2.1 is written as follows.

Problem 3.1. Find $(\boldsymbol{u}_h^n,p_h^n)\in U_h\times M_h$ $(1\le n\le N)$ such that

$$\begin{cases} (\bar{\partial}_t\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)+a(\boldsymbol{u}_h^n,\boldsymbol{v}_h)+a_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)-b(\boldsymbol{v}_h,p_h^n)\\ =(\boldsymbol{f}^n,\Pi_h^*\boldsymbol{v}_h),\quad \forall\boldsymbol{v}_h\in U_h,\quad n=1,2,\cdots,N,\\ b(\boldsymbol{u}_h^n,q_h)=0,\quad \forall q_h\in M_h,\quad n=1,2,\cdots,N,\\ \boldsymbol{u}_h^0=\Pi_h\boldsymbol{u}_0,\quad (x,y)\in\Omega, \end{cases} \tag{3.19}$$

where

$$a_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)=((\boldsymbol{u}_h^{n-1}\cdot\nabla)\boldsymbol{u}_h^n,\boldsymbol{\Pi}_h^*\boldsymbol{v}_h)+((\mathrm{div}\boldsymbol{u}_h^{n-1})\boldsymbol{u}_h^n,\boldsymbol{\Pi}_h^*\boldsymbol{v}_h)/2. \tag{3.20}$$

Remark 3.1. Problem 3.1 is also referred to as Euler backward one step fully discrete FVE formulation. The trilinear form $a_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)$ satisfies (see [20,21])

$$\begin{aligned} &|a_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)|\\ \le&\tilde{C}_2\big(|\boldsymbol{u}_h^{n-1}|_1\|\boldsymbol{u}_h^n\|_0^{\frac12}|\boldsymbol{u}_h^n|_1^{\frac12}+\|\boldsymbol{u}_h^{n-1}\|_0^{\frac12}|\boldsymbol{u}_h^{n-1}|_1^{\frac12}|\boldsymbol{u}_h^n|_1\big)\|\boldsymbol{v}_h\|_0,\quad \forall\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h\in U_h. \end{aligned} \tag{3.21}$$

4 Existence and uniqueness, stability, and error estimate of the fully discrete FVE solutions for Problem IV

In order to derive the existence and uniqueness, the stability, and the error estimates of the fully discrete FVE solutions for non-stationary Navier-Stokes equations, it is necessary to introduce the following discrete Gronwall Lemma (see [22,23]).

Lemma 4.1 (Discrete Gronwall Lemma)**.** *If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three positive sequence, and $\{c_n\}$ is monotone, they satisfy*

$$a_n+b_n\le c_n+\bar{\lambda}\sum_{i=0}^{n-1}a_i,\quad \bar{\lambda}>0,\quad a_0+b_0\le c_0, \tag{4.1}$$

then

$$a_n+b_n\le c_n\exp(n\bar{\lambda}),\quad n\ge 0. \tag{4.2}$$

By the standard FE method for non-stationary Navier-Stokes equations (see [1–3, 22, 23]), we have the following lemma, whose proof is provided in Appendix A.

Lemma 4.2. *Let $(S_h\boldsymbol{u}^n, Q_h p^n)$ be the Navier-Stokes projection of the solutions $(\boldsymbol{u}^n, p^n)$ for Problem 2.2 on $U_h\times M_h$, i.e., for the solutions $(\boldsymbol{u}^n,p^n)\in U\times M$ for Problem 2.2, there exist $(S_h\boldsymbol{u}^n,Q_hp^n)$ $(n=1,2,\cdots,N)$ such that*

$$\begin{aligned}
&k\mathcal{A}_h((S_h\boldsymbol{u}^n,Q_hp^n);(\boldsymbol{v}_h,q_h))+(S_h\boldsymbol{u}^n-S_h\boldsymbol{u}^{n-1},\boldsymbol{v}_h)=k\mathcal{A}((\boldsymbol{u}^n,p^n);(\boldsymbol{v}_h,q_h))\\
&\quad+(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{v}_h),\quad \forall(\boldsymbol{v}_h,q_h)\in U_h\times M_h,\quad n=1,2,\cdots,N, \qquad (4.3a)\\
&S_h\boldsymbol{u}^0=\Pi_h\boldsymbol{u}_0(x,y),\quad \boldsymbol{u}^0=\boldsymbol{u}_0(x,y),\quad (x,y)\in\Omega, \qquad (4.3b)
\end{aligned}$$

where $\mathcal{A}_h((S_h\boldsymbol{u}^n,p_h^n);(\boldsymbol{v}_h,q_h))=a(S_h\boldsymbol{u}^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,Q_hp^n)+b(S_h\boldsymbol{u}^n,q_h)+a_1(S_h\boldsymbol{u}^{n-1},S_h\boldsymbol{u}^n,\boldsymbol{v}_h)$, $\mathcal{A}((\boldsymbol{u}^n,p^n);(\boldsymbol{v}_h,q_h))=a(\boldsymbol{u}^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,p^n)+b(\boldsymbol{u}^n,q_h)+a_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)$.

If the solution $(\boldsymbol{u}^n,p^n)\in H^2(\Omega)^2\times H^1(\Omega)$ $(n=1,2,\cdots,N)$ for Problem 2.2, then there hold the following error estimates

$$\|\boldsymbol{u}^n-S_h\boldsymbol{u}^n\|_0^2+k\nu\sum_{i=1}^n\|\boldsymbol{u}^i-S_h\boldsymbol{u}^i\|_1^2\le Ch^4,\quad \|p^n-Q_hp^n\|_0\le Ch. \qquad (4.4)$$

There hold the existence, uniqueness, and stability of solutions for Problem 3.1.

Theorem 4.1. *Under the assumptions of Theorem 2.1, there exists a unique series of solutions $(\boldsymbol{u}_h^n,p_h^n)$ $(n=1,2,\cdots,N)$ to fully discrte FVE formulation Problem 3.1 satisfying the following stability*

$$\|\boldsymbol{u}_h^n\|_0^2+k^2\|p_h^n\|_0^2+k\sum_{i=1}^n\|\boldsymbol{u}_h^i\|_1^2\le C\big(\|\boldsymbol{u}_0\|_0^2+\|\boldsymbol{f}\|^2_{L^\infty(H^{-1})}\big). \qquad (4.5)$$

Proof. Let $\tilde a(\boldsymbol{u}_h^n,\boldsymbol{v}_h)=(\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)+ka(\boldsymbol{u}_h^n,\boldsymbol{v}_h)+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)$, $\tilde F(\boldsymbol{v}_h)=(\boldsymbol{u}_h^{n-1},\Pi_h^*\boldsymbol{v}_h)+k(\boldsymbol{f}^n,\Pi_h^*\boldsymbol{v}_h)$. Then Problem 3.1 can be rewritten as

$$\begin{cases}
\tilde a(\boldsymbol{u}_h^n,\boldsymbol{v}_h)-kb(\boldsymbol{v}_h,p_h^n)=\tilde F(\boldsymbol{v}_h), & \forall\boldsymbol{v}_h\in U_h,\quad n=1,2,\cdots,N,\\
b(\boldsymbol{u}_h^n,q_h)=0, & \forall q_h\in M_h,\quad n=1,2,\cdots,N,\\
\boldsymbol{u}_h^0=\Pi_h\boldsymbol{u}_0, & (x,y)\in\Omega.
\end{cases} \qquad (4.6)$$

For given $\boldsymbol{u}_h^{n-1}\in U_n$ and $\boldsymbol{f}^n\in H^{-1}(\Omega)^2$, it is obvious that $\tilde a(\cdot,\cdot)$ is the bounded bilinear form and $\tilde F(\cdot,\cdot)$ the bounded linear form. If $\boldsymbol{u}_h^{n-1}\in U_n$ is bounded, for example, $\|\nabla\boldsymbol{u}_h^{n-1}\|_0\le N_0^{-1}\nu/2$ (Due to the solution for Problem 2.1 satisfying a similar condition), from (2.2), (3.21), and Lemma 3.3, we have

$$\begin{aligned}
&\tilde a(\boldsymbol{u}_h^n,\boldsymbol{u}_h^n)=(\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{u}_h^n)+ka(\boldsymbol{u}_h^n,\boldsymbol{u}_h^n)+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{u}_h^n)\\
&\ge|||\boldsymbol{u}_h^n|||_0^2+k\nu\|\nabla\boldsymbol{u}_h^n\|_0^2-k\tilde C_2N_0\nu\|\nabla\boldsymbol{u}_h^n\|_0|||\boldsymbol{u}_h^n|||_0/2\\
&\ge(1-0.25k\tilde C_2^2N_0^2)|||\boldsymbol{u}_h^n|||_0^2+0.75k\nu\|\nabla\boldsymbol{u}_h^n\|_0^2\ge\tilde\alpha\|\boldsymbol{u}_h^n\|_1^2,\quad \forall\boldsymbol{u}_h^n\in U_h, \qquad (4.7)
\end{aligned}$$

where $\tilde{\alpha}=\min\{(1-0.25k\tilde{C}_2^2N_0^2)C_3^2,0.75k\nu\}$. Thus, if k is sufficiently small such that $0.25k\tilde{C}_2^2N_0^2<1$, the bilinear form $\tilde{a}(\cdot,\cdot)$ is coercive. In addition, $kb(\boldsymbol{v}_h,p_h^n)$ also satisfies discrete B-B inequality. There exists a unique series of solutions $(\boldsymbol{u}_h^n,p_h^n)$ $(n=1,2,\cdots,N)$ for (4.6), i.e., Problem 3.1 based on saddle theorem of mixed FE methods (see [1–3,23,24]). Taking $\boldsymbol{v}_h=\boldsymbol{u}_h^n$ and $q_h=p_h^n$ in Problem 3.1, by using Lemmas 3.2-3.4, (3.21), Hölder inequality, and Cauchy inequality, we obtain that

$$
\begin{aligned}
&|||\boldsymbol{u}_h^n|||_0^2+k\nu|\boldsymbol{u}_h^n|_1^2=k(\boldsymbol{f}^n,\Pi_h^*u_h^n)+(\boldsymbol{u}_h^{n-1},\Pi_h^*u_h^n)+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{u}_h^n)\\
\le&Ck\nu^{-1}\|\boldsymbol{f}^n\|_{-1}^2+\frac{k\nu}{2}|\nabla\boldsymbol{u}_h^n|_0^2+\frac{1}{2}|||\boldsymbol{u}_h^{n-1}|||_0^2+\frac{1}{2}|||\boldsymbol{u}_h^n|||_0^2+\frac{kC}{4}|||\boldsymbol{u}_h^n|||_0^2.
\end{aligned}\tag{4.8}
$$

Further, we get from (4.8) that

$$
|||\boldsymbol{u}_h^n|||_0^2+k\nu|\boldsymbol{u}_h^n|_1^2\le Ck\nu^{-1}\|\boldsymbol{f}^n\|_{-1}^2+|||\boldsymbol{u}_h^{n-1}|||_0^2+\frac{kC}{2}|||\boldsymbol{u}_h^n|||_0^2.\tag{4.9}
$$

Summing (4.9) from 1 to n and noting that $\boldsymbol{u}_h^0=\Pi_h\boldsymbol{u}_0$, we have that

$$
\|\boldsymbol{u}_h^n\|_0^2+k\nu\sum_{i=1}^n|\boldsymbol{u}_h^i|_1^2\le C\Big(\|\boldsymbol{u}_0\|_0^2+k\nu^{-1}\sum_{i=1}^n\|\boldsymbol{f}^n\|_{-1}^2+k\sum_{i=0}^n\|\boldsymbol{u}_h^i\|_0^2\Big).\tag{4.10}
$$

If k is sufficiently small such that $Ck\le 1/2$ in (4.10), it can be reduced that

$$
\|\boldsymbol{u}_h^n\|_0^2+k\nu\sum_{i=1}^n|\boldsymbol{u}_h^i|_1^2\le C\Big(\|\boldsymbol{u}_0\|_0^2+k\nu^{-1}\sum_{i=1}^n\|\boldsymbol{f}^n\|_{-1}^2+k\sum_{i=0}^{n-1}\|\boldsymbol{u}_h^i\|_0^2\Big).\tag{4.11}
$$

Applying discrete Gronwall Lemma 4.1 to (4.11) yields

$$
\|\boldsymbol{u}_h^n\|_0^2+k\nu\sum_{i=1}^n|\boldsymbol{u}_h^i|_1^2\le C\big(\|\boldsymbol{u}_0\|_0^2+k\nu^{-1}\sum_{i=1}^n\|\boldsymbol{f}^n\|_{-1}^2\big)\exp(Ck)\le C\big(\|\boldsymbol{u}_0\|_0^2+\|\boldsymbol{f}\|_{L^\infty(H^{-1})}^2\big).\tag{4.12}
$$

Using Problem 3.1, (3.18), (2.5) and Hölder inequality yields that

$$
\begin{aligned}
&\beta k\|p_h^n\|_0\le\sup_{\boldsymbol{v}_h\in U_h}\frac{k|b_h(\boldsymbol{v}_h,p_h^n)|}{\|\boldsymbol{v}_h\|_1}\\
=&\sup_{\boldsymbol{v}_h\in U_h}[|(\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)+ka_h(\boldsymbol{u}_h^n,\boldsymbol{v}_h)|+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)\\
&\quad-k(\boldsymbol{f}^n,\Pi_h^*\boldsymbol{v}_h)-(\boldsymbol{u}_h^{n-1},\Pi_h^*\boldsymbol{v}_h)|]/\|\boldsymbol{v}_h\|_1\\
\le&C\big(\|\boldsymbol{u}_h^n\|_0+k\nu\|\boldsymbol{u}_h^n\|_1+kN_0\|\boldsymbol{u}_h^{n-1}\|_1\|\boldsymbol{u}_h^n\|_1+k\|\boldsymbol{f}^n\|_{-1}+\|\boldsymbol{u}_h^{n-1}\|_0\big)\\
\le&C(\|\boldsymbol{u}_0\|_0+\|\boldsymbol{f}\|_{L^\infty(H^{-1})}).
\end{aligned}\tag{4.13}
$$

Combining (4.13) and (4.12) yields (4.5), which completes the proof of Theorem 4.1. □

Remark 4.1. The inequality (4.5) in Theorem 4.1 shows that the solutions of Problem 3.1 is bounded, stabilized, and continuously depending on the initial value $\boldsymbol{u}_0$ and the body force term $\boldsymbol{f}$.

There are the following results of convergence, i.e., error estimates of the solutions $(\boldsymbol{u}_h^n,p_h^n)$ $(n=1,2,\cdots,N)$ for Problem 3.1.

Theorem 4.2. *Under the hypotheses of Theorem 2.1, if the solutions* $(\boldsymbol{u}_h^n,p_h^n)$ *of Problem 3.1 satisfy* $|\boldsymbol{u}_h^n|_1\le N_0^{-1}\nu/2$ $(n=1,2,\cdots,N)$, *then there hold the following error estimates between the solutions* $(\boldsymbol{u}^n,p^n)$ $(n=1,2,\cdots,N)$ *to Problem 2.2 and the FVE solutions* $(\boldsymbol{u}_h^n,p_h^n)$ $(n=1,2,\cdots,N)$ *to Problem 3.1,*

$$\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|_0^2+k\nu\sum_{i=1}^{n}|\boldsymbol{u}^i-\boldsymbol{u}_h^i|_1^2\le C(h^4+k^2), \tag{4.14a}$$

$$\|p^n-p_h^n\|_0\le C(h+k). \tag{4.14b}$$

Proof. Subtracting Problem 3.1 from Problem 2.2 taking $\boldsymbol{v}=\boldsymbol{v}_h$ and $q=q_h$, and using Lemmas 3.1, 3.3 and 3.4 yield that the system of error equations

$$\begin{cases} (\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{v}_h)+(\boldsymbol{u}_h^n-\Pi_h^*\boldsymbol{u}_h^n,\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h)+ka(\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{v}_h)\\ +ka_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)-ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)-kb(\boldsymbol{v}_h,p^n-p_h^n)\\ =k(\boldsymbol{f}^n-\Pi_h^*\boldsymbol{f}^n,\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h)+(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h)\\ +(\boldsymbol{u}_h^{n-1}-\Pi_h^*\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h),\quad \forall \boldsymbol{v}_h\in \boldsymbol{U}_h,\quad n=1,2,\cdots,N,\\ b(\boldsymbol{u}^n-\boldsymbol{u}_h^n,q_h)=0,\quad \forall q_h\in M_h,\quad n=1,2,\cdots,N,\\ \boldsymbol{u}^0-\boldsymbol{u}_h^0=\boldsymbol{u}_0(x,y)-\Pi_h\boldsymbol{u}_0(x,y),\quad (x,y)\in\Omega. \end{cases} \tag{4.15}$$

Put $\boldsymbol{e}^n=S_h\boldsymbol{u}^n-\boldsymbol{u}_h^n$. By using the system of error equations (4.15), (4.3a), and Lemmas 3.3 and 3.4, we obtain that

$$\begin{aligned} &\|\boldsymbol{e}^n\|_0^2+k\nu|\boldsymbol{e}^n|_1^2=(\boldsymbol{e}^n,\boldsymbol{e}^n)+ka(\boldsymbol{e}^n,\boldsymbol{e}^n)\\ =&(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\boldsymbol{e}^n)+ka(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\boldsymbol{e}^n)+(\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{e}^n)+ka(\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{e}^n)\\ =&(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\boldsymbol{e}^n)+ka(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\boldsymbol{e}^n)+kb(\boldsymbol{e}^n,p^n-p_h^n)-ka_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{e}^n)\\ &+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{e}^n)+(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{e}^n)-(\boldsymbol{u}_h^n-\Pi_h^*\boldsymbol{u}_h^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)\\ &+(\boldsymbol{u}_h^{n-1}-\Pi_h^*\boldsymbol{u}_h^{n-1},\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)+k(\boldsymbol{f}^n-\Pi_h^*\boldsymbol{f}^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)\\ =&(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\boldsymbol{e}^n)+ka(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\boldsymbol{e}^n)-kb(\boldsymbol{e}^n,Q_hp^n-p^n)+(\boldsymbol{e}^{n-1},\boldsymbol{e}^n)\\ &-ka_1(\boldsymbol{u}^{n-1},\boldsymbol{u}^n,\boldsymbol{e}^n)+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{e}^n)+(\boldsymbol{u}^{n-1}-S_h\boldsymbol{u}^{n-1},\boldsymbol{e}^n)\\ &+k(\boldsymbol{f}^n-\Pi_h^*\boldsymbol{f}^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)-(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}-\Pi_h^*(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}),\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)\\ =&k(\boldsymbol{f}^n-\Pi_h^*\boldsymbol{f}^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)-(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}-\Pi_h^*(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}),\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)\\ &+(\boldsymbol{e}^{n-1},\boldsymbol{e}^n)+ka_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)-ka_1(\boldsymbol{e}^{n-1},\boldsymbol{u}_h^n,\boldsymbol{e}^n). \end{aligned} \tag{4.16}$$

Using (3.9), (4.4), Hölder inequality, and Cauchy inequality, we have

$$|k(\boldsymbol{f}^n-\Pi_h^*\boldsymbol{f}^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)|\le Ckh^2\|\boldsymbol{f}^n\|_1|\boldsymbol{e}^n|_1\le Ckh^4\|\boldsymbol{f}^n\|_1^2+\frac{\nu k}{8}|\boldsymbol{e}^n|_1^2, \tag{4.17a}$$

$$|(\boldsymbol{e}^{n-1},\boldsymbol{e}^n)|\le\frac{1}{2}\left(\|\boldsymbol{e}^{n-1}\|_0^2+\|\boldsymbol{e}^n\|_0^2\right). \tag{4.17b}$$

With inverse error estimate theory and Taylor's formula, we have that

$$\begin{aligned}&|(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}-\Pi_h^*(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}),\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)|\le Ch^2|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}|_1|\boldsymbol{e}^n|_1^2\\ \le& Ch^3|\boldsymbol{e}^n|_1^2+Ch^3\|\nabla(S_h\boldsymbol{u}^n-S_h\boldsymbol{u}^{n-1})\|_0^2+Ch^3|\boldsymbol{e}^{n-1}|_1^2+\frac{k\nu}{4}|\boldsymbol{e}^n|_1^2\\ \le& Ch\|\boldsymbol{e}^{n-1}\|_0^2+Ck^2h^3\|u_t\|_{L^\infty(H^1)}^2+Ch\|\boldsymbol{e}^n\|_0^2+\frac{k\nu}{8}|\boldsymbol{e}^n|_1^2.\end{aligned} \tag{4.18}$$

If $|\boldsymbol{u}_h^n|_1\le N_0^{-1}\nu/2$ $(n=0,1,2,\cdots,N)$, by using (2.2), (2.5), (3.9), (3.21) and Lemma 3.1, we obtain that

$$k|a_{1h}(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{e}^n-\Pi_h^*\boldsymbol{e}^n)-a_1(\boldsymbol{e}^{n-1},\boldsymbol{u}_h^n,\boldsymbol{e}^n)|\le Ckh^4+\frac{k\nu}{4}|\boldsymbol{e}^n|_1^2. \tag{4.19}$$

Combining (4.17a)-(4.19) and (4.16), we obtain

$$\|\boldsymbol{e}^n\|_0^2+k\nu|\boldsymbol{e}^n|_1^2\le Ckh^4+Ck^2h^3+\|\boldsymbol{e}^{n-1}\|_0^2+Ch\|\boldsymbol{e}^{n-1}\|_0^2+Ch\|\boldsymbol{e}^n\|_0^2. \tag{4.20}$$

Summing (4.20) from 1 to n, if h is sufficiently small such that $Ch\le 1/2$ in (4.20), we have

$$\|\boldsymbol{e}^n\|_0^2+2k\nu\sum_{i=1}^n|\boldsymbol{e}^i|_1^2\le Cnkh^4+Cnk^2h^3+2\|\boldsymbol{e}^0\|_0^2+Ch\sum_{i=0}^{n-1}\|\boldsymbol{e}^i\|_0^2. \tag{4.21}$$

Due to Lemma 4.2 and (3.2), $\|S_h\boldsymbol{u}^0-\boldsymbol{u}^0\|_0^2+\|\boldsymbol{u}_0-\boldsymbol{u}_h^0\|_0^2\le Ch^4$. By using Gronwall Lemma 4.1 and Cauchy inequality, we obtain

$$\|\boldsymbol{e}^n\|_0^2+k\nu\sum_{i=1}^n|\boldsymbol{e}^i|_1^2\le C(h^4+kh^3)\exp(Cnh)\le C(h^4+k^2). \tag{4.22}$$

Using (4.4) and triangle inequality yields that

$$\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|_0^2+k\nu\sum_{i=1}^n|\boldsymbol{u}^i-\boldsymbol{u}_h^i|_1^2\le C(h^4+k^2). \tag{4.23}$$

From (3.18), (2.2), error equation (4.15), and Lemma 4.2, we have that

$$
\begin{aligned}
&\beta\|Q_h p^n - p_h^n\|_0 \le \sup_{v_h\in U_h}\frac{|b(v_h,Q_h p^n - p_h^n)|}{\|v_h\|_1}\\
&\le \sup_{v_h\in U_h}\frac{|b(v_h,Q_h p^n - p^n)|}{\|v_h\|_1} + \sup_{v_h\in U_h}\frac{|b(v_h,p^n - p_h^n)|}{\|v_h\|_1}\\
&\le C\|Q_h p^n - p^n\|_0 + \big|k^{-1}\big[(u^n-u_h^n,v_h)+(u_h^n-\Pi_h^* u_h^n, v_h-\Pi_h^* v_h)\\
&\quad -(u^{n-1}-u_h^{n-1},v_h)-(u_h^{n-1}-\Pi_h^* u_h^{n-1},v_h-\Pi_h^* v_h)+a(u^n-u_h^n,v_h)\big]\\
&\quad +a_1(u^{n-1},u^n-u_h^n,v_h)+a_1(u^{n-1}-u_h^{n-1},u_h^n,v_h)\\
&\quad +a_{1h}(u_h^{n-1},u_h^n,v_h-\Pi_h^* v_h)-(f^n-\Pi_h^* f^n,v_h-\Pi_h^* v_h)\big|/\|v_h\|_1\\
&\le C\big[\|Q_h p^n - p^n\|_0 + k^{-1}\big(\|u^n-u_h^n\|_0+\|u^{n-1}-u_h^{n-1}\|_0\big)\\
&\quad +|u^n-u_h^n|_1+|u^{n-1}-u_h^{n-1}|_1+h^2\|u_h^n-u_h^{n-1}\|_1+Ch^2\|f^n\|_1\big]\\
&\le C(h+k).
\end{aligned}
\tag{4.24}
$$

Applying triangle inequality and Lemma 4.2 yields that

$$\|p^n - p_h^n\|_0 \le C(h+k), \tag{4.25}$$

which completes Theorem 4.2. □

Combining Theorems 2.1 and 4.2 yields the following results.

Theorem 4.3. *Under the hypotheses of Theorems 2.1 and 4.2, there hold the error estimates between the solution (u,p) to Problem 2.1 and the solutions (u_h^n,p_h^n) $(n=1,2,\cdots,N)$ to Problem 3.1:*

$$\|u(t_n)-u_h^n\|_0 + k\sum_{i=1}^{n}\|\nabla(u(t_i)-u_h^i)\|_0 \le C(k+h^2), \tag{4.26a}$$

$$\|p(t_n)-p_h^n\|_0 \le C(k+h), \tag{4.26b}$$

where C is a constant which is only dependent on ν, Ω, maximum total time upper bound T, u_0, and the force term f, but independent of k and h.

5 Some numerical experiments

In this section, we present some numerical experiments with a physical model of cavity flow of Reynolds number $Re=10^3$ by the fully discrete FVE Formulation, i.e., Problem 3.1 to validate that the results of numerical computation are consistent with the theoretical conclusions. Moreover, it is shown that FVE method is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations and that

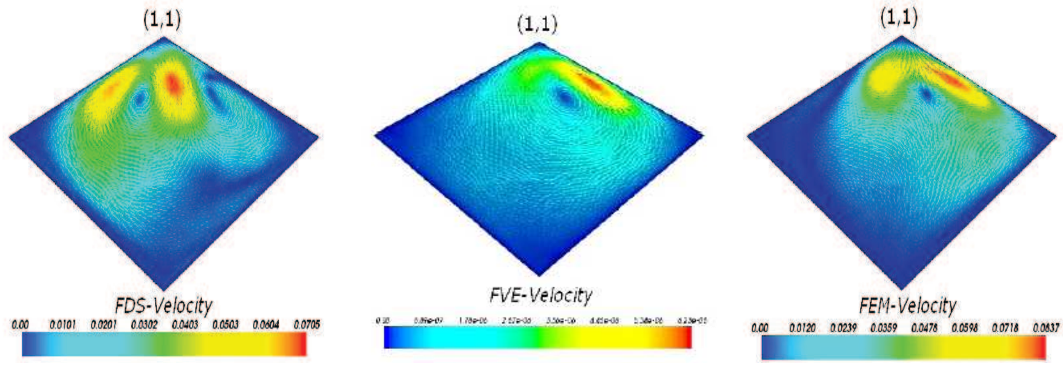

Figure 2: When $Re=10^3$, the left chart, the center chart, the right char are respectively the stream line figures of the FD solution, the FVE solution, and the FE solution of the velocity $\boldsymbol{u}$ at the time level $t=2$.

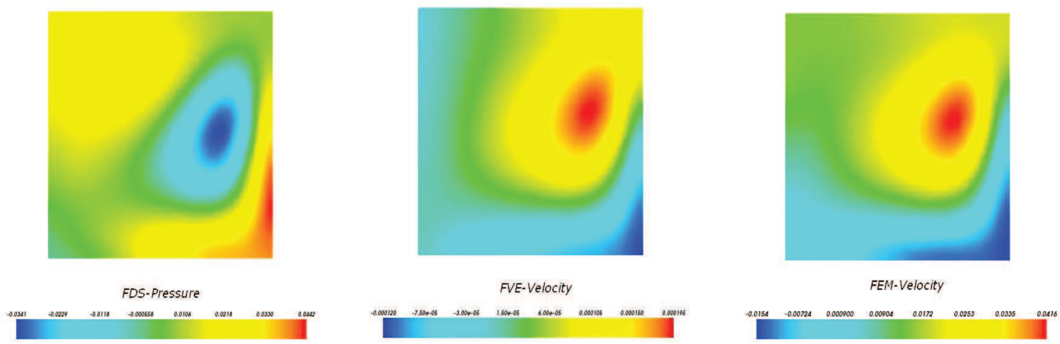

Figure 3: When $Re=10^3$, the left chart, the center chart, the right chart are respectively the stream line figures of the FD solution, the FVE solution, and the FE solution of the pressure p at the time level $t=2$.

FVE formulation is one of the most effective numerical methods by comparing the FVE solutions with the FE solutions and the FD solutions for the non-stationary Navier-Stokes equations.

Let the side length of the cavity be 1 and $\overline{\Omega}=[0,1]\times[0,1]$. We first divide the cavity into $100\times 100=10000$ small squares with side l ength $\Delta x=\Delta y=0.01$, and then link diagonal of the square to divide each square into two triangles in the same direction, which composes triangularizations $\Im_h$ $(h=\sqrt{2}\times 10^{-2})$ and combines into $\Im_{2h}$. The dual decomposition $\Im_h^*$ is taken as barycenter dual decomposition, i.e., the barycenter of the right triangle $K\in\Im_h$ is taken as the node of the dual decomposition. We take a time step increment as $k=0.01$. Let the initial value and the boundary value of $\boldsymbol{u}=(u_1,u_2)$ be $\boldsymbol{u}_0(x,y)=\boldsymbol{\varphi}(x,y,t)=(yx,0)$. And let $\boldsymbol{f}=\boldsymbol{0}$ on $\bar{\Omega}$.

We find a FVE numerical solution $(\boldsymbol{u}_h^n,p_h^n)$ by Problem 3.1 when $n=200$ (i.e., $t=2$), whose $\boldsymbol{u}_h^n$ and p_h^n are depicted graphically at the center charts in Figs. 2 and 3, respectively.

We also find a numerical FE solution $(\boldsymbol{u}_h^n,p_h^n)$ when $n=200$ (i.e., $t=2$) by the following fully discrete FE formulation:

$$\begin{cases} (\boldsymbol{u}_h^n,\boldsymbol{v}_h)+ka(\boldsymbol{u}_h^n,\boldsymbol{v}_h)+ka_1(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h)-kb(p_h^n,\boldsymbol{v}_h)=(\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h\in U_h,\\ b(q_h,\boldsymbol{u}_h^n)=0, \quad \forall q_h\in M_h, \quad i=1,2,\cdots,200,\\ \boldsymbol{u}_h^0=(x,y,0), \quad (x,y)\in\Omega, \end{cases}$$

whose $\boldsymbol{u}_h^n$ and p_h^n are depicted at the right charts in Figs. 2 and 3, respectively.

We also find numerical FD solutions $u^n_{1,j+1/2,i}$, $u^n_{2,j,i+1/2}$ and p^n_{ji} $(i,j=0,1,\cdots,100;\ n=0,1,2,\cdots,200)$ by the following FD scheme:

$$\begin{aligned} &\Big[\frac{p_{j-1,i}-2p_{j,i}+p_{j+1,i}}{\Delta x^2}+\frac{p_{j,i-1}-2p_{j,i}+p_{j,i+1}}{\Delta y^2}\Big]^n=R,\\ &u^{n+1}_{1,j+\frac12,i}=F^n_{j+\frac12,i}-\frac{k}{\Delta x}[p^n_{j+1,i}-p^n_{j,i}], \quad u^{n+1}_{2,j,i+\frac12}=G^n_{j,i+\frac12}-\frac{k}{\Delta y}[p^n_{j,i+1}-p^n_{j,i}], \end{aligned}$$

where

$$\begin{aligned} R=&\frac{1}{k\Delta x}\big[F_{j+\frac12,i}-F_{j-\frac12,i}\big]^n+\frac{1}{k\Delta y}\big[G_{j,i+\frac12}-G_{j,i-\frac12}\big]^n,\\ F^n_{j+\frac12,i}=&u^n_{1,j+\frac12,i}-\frac{k}{\Delta x}u^n_{1,j+\frac12,i}(u^n_{1,j+1,i}-u^n_{1,j,i})-\frac{k}{\Delta y}u^n_{2,j+\frac12,i}(u^n_{1,j+\frac12,i+\frac12}-u^n_{1,j+\frac12,i-\frac12})\\ &+\nu k\Big[\frac{u_{1,j+\frac12,i-1}-2u_{1,j+\frac12,i}+u_{1,j+\frac12,i+1}}{\Delta y^2}+\frac{u_{1,j-\frac12,i}-2u_{1,j+\frac12,i}+u_{1,j+\frac32,i}}{\Delta x^2}\Big]^n,\\ G^n_{j,i+\frac12}=&u^n_{2,j,i+\frac12}-\frac{k}{\Delta y}u^n_{2,j,i+\frac12}(u^n_{2,j,i+1}-u^n_{2,j,i})-\frac{k}{\Delta x}u^n_{1,j,i+\frac12}(u^n_{2,j+\frac12,i+\frac12}-u^n_{2,j-\frac12,i+\frac12})\\ &+\nu k\Big[\frac{u_{2,j-1,i+\frac12}-2u_{2,j,i+\frac12}+u_{2,j+1,i+\frac12}}{\Delta x^2}+\frac{u_{2,j,i-\frac12}-2u_{2,j,i+\frac12}+u_{2,j,i+\frac32}}{\Delta y^2}\Big]^n, \end{aligned}$$

when $n=200$ (i.e., $t=2$), whose $(u^n_{1,j+1/2,i},u^n_{2,j,i+1/2})$ and $p^n_{i,j}$ are depicted graphically at the left charts in Figures 2 and 3, respectively.

The curves of the left chart, the center chart, the right char in Fig. 4 are respectively the relative errors of the FVE solutions, the FD solutions, and the FE solutions at time $t\in(0,2]$. Since the fully discrete FVE formulation Problem 3.1 keeps conservation law of mass or energy, it is more stable than the FD scheme and the fully discrete FE formulation and the errors of its numerical solutions are smallest among three formulations, which does not exceed 4×10^{-2}. Moreover, it is shown that the results for numerical examples are consistent with those obtained for the theoretical case.

Comparing the FVE solutions with the FD solutions and the FE solutions implementing numerical simulation for $t=2$, we find that for the fully discrete FVE formulation Problem 3.1 the required computing time is about 6 minutes, while for the fully discrete FE formulation is about 12 minutes, and FD scheme is about 5 minutes, computing time of the fully discrete FVE formulation Problem 3.1 is almost as same as the times of FD

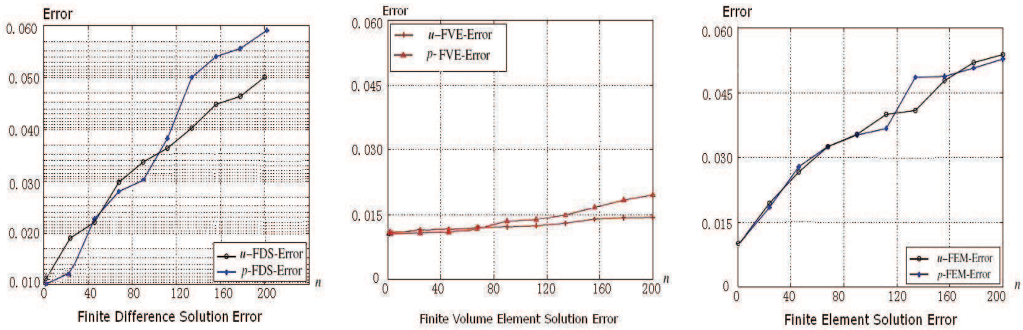

Figure 4: When $Re=10^3$, the left chart, the center chart, the right char are respectively the relative error figures of the FD solutions, the FVE solutions, and the FE solution of the velocity $\boldsymbol{u}$ and the pressure p at the time $t\in(0,2]$.

scheme, but is only a half of time of the FE formulation and the computing error accumulations of FVE solutions are far smaller than those of the FE solutions and the FD solutions.

It has been shown that FVE formulation is one of the most effective numerical methods by comparing the above results of the numerical simulations of FVE formulation with those of FE formulation and FD scheme for the non-stationary Navier-Stokes equations.

6 Conclusions and discussion

In this paper, we have first derived the semi-discrete formulation with respect to time for the non-stationary Navier-Stokes equations. Next, we have directly established the new fully discrete FVE formulation based on macroelement from the semi-discrete formulation with respect to time. Then, we have provided the error estimates between the fully discrete FVE solutions and the accuracy solution by means of the standard FE method for the non-stationary Navier-Stokes equations. Finally, we have provided some numerical experiments to validate that the numerical errors between the fully discrete FVE solutions and the accuracy solution are consistent with the theoretical results obtained previously, that FVE method is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations, and that the FVE formulation is one of the most effective numerical methods by comparing the results of the numerical results of the FVE formulation with those of the FE formulation and the FD scheme for the non-stationary Navier-Stokes equations.

Although some stabilized or penalty FVE methods for the non-stationary Navier-Stokes equations have been presented, the FVE formulation here is a direct discrete method (without any stabilization and penalty) based on macroelement and the discrete B-B inequality satisfied which includes more generality than those of stabilized or penalty FVE formulations. Moreover, though some stabilization or penalty FVE formulations have better stabilization than those without any stabilization, they are easy to cause distortion from their accuracy solutions, i.e., are easy to deviate the original solutions. Therefore, making the study of the FVE method without any stabilization for the non-stationary Navier-Stokes equations has far more important and more serviceable than those with stabilizations. In addition, the research method here is directly to establish the fully discrete FVE formulation from the semi-discrete formulation with respect to time and to do theoretical analysis which avoids the semi-discrete FVE formulation with respect to space variable, that is it is unnecessary to discuss the semi-discrete FVE formulation with respect to space variable. These are the improvement and innovation for the existing methods which is a new study attempt.

Appendix

The proof of Lemma 4.2 is as follows.

Let $\tilde{A}(\tilde{\boldsymbol{u}}_h^n,\boldsymbol{v}_h)=a(\tilde{\boldsymbol{u}}_h^n,\boldsymbol{v}_h)+a_1(\tilde{\boldsymbol{u}}_h^{n-1},\tilde{\boldsymbol{u}}_h^n,\boldsymbol{v}_h)+(\tilde{\boldsymbol{u}}_h^n,\boldsymbol{v}_h)/k$ and $\bar{F}(\boldsymbol{v}_h)=(\tilde{\boldsymbol{u}}_h^{n-1},\boldsymbol{v}_h)/k+(\boldsymbol{f}^n,\boldsymbol{v}_h)$. Then mixed FE formulation for Problem 2.1 is follows

$$\begin{cases} \tilde{A}(\tilde{\boldsymbol{u}}_h^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,\tilde{p}_h^n)=\bar{F}(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h\in U_h, \quad n=1,2,\cdots,N, \\ b(\tilde{\boldsymbol{u}}_h^n,q_h)=0, & \forall q_h\in M_h, \quad n=1,2,\cdots,N, \\ \tilde{\boldsymbol{u}}_h^0=\Pi_h\boldsymbol{u}_0, & (x,y)\in\Omega. \end{cases} \tag{A.1}$$

For given $\tilde{\boldsymbol{u}}_h^{n-1}\in U_n$ and $\boldsymbol{f}^n\in H^{-1}(\Omega)^2$, it is obvious that $\tilde{A}(\cdot,\cdot)$ is the bounded bilinear form on $U_h\times U_h$ and $\bar{F}(\cdot)$ the bounded linear form on U_h. It follows from (2.2) and (2.3) that

$$\begin{aligned} \tilde{A}(\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^n) &=(\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^n)/k+a(\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^n)+a_1(\tilde{\boldsymbol{u}}_h^{n-1},\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^n) \\ &=(\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^n)/k+a(\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^n) \\ &\geq\bar{\alpha}\|\boldsymbol{u}_h^n\|_1^2, \quad \forall \boldsymbol{u}_h^n\in U_h, \end{aligned} \tag{A.2}$$

where $\bar{\alpha}=\min\{\nu,1/k\}$. Thus, the bilinear form $\tilde{A}(\cdot,\cdot)$ is coercive. In addition, U_h, M_h, and $b(\boldsymbol{v}_h,p_h^n)$ satisfy discrete B-B inequality (3.1), there exists a unique series of solutions $(\tilde{\boldsymbol{u}}_h^n,\tilde{p}_h^n)$ $(n=1,2,\cdots,N)$ for (A.1) based on saddle theorem of mixed FE methods (see [1–3, 23,24]).

Put $S_h\boldsymbol{u}^n=\tilde{\boldsymbol{u}}_h^n$ $(n=0,1,\cdots,N)$ and $Q_hp^n=\tilde{p}_h^n$ $(n=1,2,\cdots,N)$ in (A.1). Subtracting (A.1) from Problem 2.2 taking $\boldsymbol{v}=\boldsymbol{v}_h$ and $q=q_h$ yields (4.3a) and (4.3b). It easily follows from standard mixed FE error analysis (see [1–3,22,23]) that (4.4) holds, which completes the proof of Lemma 4.2.

Acknowledgements

This work is jointly supported by National Science Foundation of China (11271127) and Science Research Project of Guizhou Province Education Department (QJHKYZ[2013]207).

References

[1] R. TEMAM, Navier-Stokes Equations (3rd), New York, North–Holland, Amsterdam, 1984.

[2] V. GIRAULT AND P. A. RAVIART, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Berlin Heidelberg, Springer-Verlag, 1986.

[3] J. G. HEYWOOD AND R. RANNACHER, *Finite element approximation of the non-stationary Navier-Stokes problem, I. regularity of solutions and second order estimates for spatial discretization*, SIAM J. Numer. Anal., 19(2) (1982), pp. 275–311.

[4] J. R. CANNON AND Y. LIN, *A priori L^2 error estimates for finite-element methods for nonlinear diffusion equations with memory*, SIAM J. Numer. Anal., 27(3) (1990), pp. 595–607.

[5] Z. CAI AND S. MCCORMICK, *On the accuracy of the finite volume element method for diffusion equations on composite grid*, SIAM J. Numer. Anal., 27(3) (1990), pp. 636–655.

[6] E. SÜLI, *Convergence of finite volume schemes for Poisson's equation on nonuniform meshes*, SIAM J. Numer. Anal., 28(5) (1991), pp. 1419–1430.

[7] W. P. JONES AND K. R. MENZIEST, *Analysis of the cell-centred finite volume method for the diffusion equation*, J. Comput. Phys., 165(1) (2000), pp. 45–68.

[8] R. E. BANK AND D. J ROSE, *Some error estimates for the box methods*, SIAM J. Numer. Anal., 24(4) (1987), pp. 777–787.

[9] R. H. LI, Z. Y. CHEN AND W. WU, Generalized Difference Methods for Differential Equations-Numerical Analysis of Finite Volume Methods, Monographs and Textbooks in Pure and Applied Mathematics 226, New York, Marcel Dekker Inc., 2000.

[10] P. BLANC, R. EYMERD AND R. HERBIN, *A error estimate for finite volume methods for the Stokes equations on equilateral triangular meshes*, Numer. Methods PDEs, 20(6) (2004), pp. 907–918.

[11] J. LI AND Z. X. CHEN, *A new stabilized finite volume method for the stationary Stokes equations*, Adv. Comput. Math., 30(2) (2009), pp. 141–152.

[12] H. R. LI, Z. D. LUO AND Q. LI, *Generalized difference methods for two-dimensional viscoelastic problems*, Chinese J. Numer. Math. Appl., 29(3) (2007), pp. 251–262.

[13] X. YE, *On the relationship between finite volume and finite element methods applied to the Stokes equations*, Numer. Methods PDEs, 17(5) (2001), pp. 440–453.

[14] S. H. CHOU AND D. Y. KWAK, *A covolume method based on rotated bilinears for the generalized Stokes problem*, SIAM J. Numer. Anal., 35(2) (1998), pp. 494–507.

[15] L. H. SHEN, J. LI AND Z. X. CHEN, *Analysis of a stabilized finite volume method for the transient Stokes equations*, Int. J. Numer. Anal. Model., 6(3) (2009), pp. 505–519.

[16] M. YANG AND H. L. SONG, *A postprocessing finite volume method for time-dependent Stokes equations*, Appl. Numer. Math., 59(8) (2009), pp. 1922–1932.

[17] I. AMMARA AND C. MASSON, *Development of a fully coupled control-volume finite element method for the incompressible Navier-Stokes equations*, Int. J. Numer. Methods Fluids, 44(6) (2004), pp. 621–644.

[18] J. LI, L. H. SHEN AND Z. X. CHEN, *Convergence and stability of a stabilized finite volume method for the stationary Navier-Stokes equations*, BIT Numerical Mathematics, 50(4) (2010), pp. 823–842.

[19] G. L. HE, Y. N. HE AND X. L. FENG, *Finite volume method based on stabilized finite elements for the nonstationary Navier-Stokes problem*, Numer. Methods PDEs, 23(5) (2007), pp. 1167–1191.

[20] G. L. HE, Y. N. HE AND Z. X. CHEN, *A penalty finite volume method for the transient Navier-Stokes equations*, Appl. Numer. Math., 58(11) (2008), pp. 1583–1613.

[21] G. HE AND Y. HE, *The finite volume method based on stabilized finite element for the stationary Navier-Stokes equations*, J. Comput. Appl. Math., 205(1) (2007), pp. 651–665.

[22] R. A. ADAMS, Sobolev Spaces, New York, Academic Press, 1975.
[23] Z. D. LUO, The Foundations and Applications of Mixed Finite Element Methods, Beijing, Science Press, 2006, in Chinese.
[24] F. BREZZI AND M. FORTIN, Mixed and Hybrid Finite Element Methods, New York, Springer-Verlag, 1991.
[25] P. G. CIARLET, The Finite Element Method for Elliptic Problems, North–Holland, Amsterdam, 1978.
[26] R. STENBERG, *Analysis of mixed finite elements for the Stokes problem: a unified approach*, Math. Comput., 42 (1984), pp. 9–23.