Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **6**, No. 5, pp. 663-679

DOI: 10.4208/aamm.2013.m153
October 2014

Two-Scale Picard Stabilized Finite Volume Method for the Incompressible Flow

Jianhong Yang[1,*], Gang Lei[1] and Jianwei Yang[2]

[1] *Department of Mathematics, Baoji University of Arts and Sciences, Baoji 721013, China*
[2] *Department of Computer and Science, Baoji University of Arts and Sciences, Baoji 721007, China*

Received 12 March 2013; Accepted (in revised version) 26 September 2013

Available online 23 July 2014

Abstract. In this paper, we consider a two-scale stabilized finite volume method for the two-dimensional stationary incompressible flow approximated by the lowest equal-order element pair P_1-P_1 which do not satisfy the inf-sup condition. The two-scale method consist of solving a small non-linear system on the coarse mesh and then solving a linear Stokes equations on the fine mesh. Convergence of the optimal order in the H^1-norm for velocity and the L^2-norm for pressure are obtained. The error analysis shows there is the same convergence rate between the two-scale stabilized finite volume solution and the usual stabilized finite volume solution on a fine mesh with relation $h=\mathcal{O}(H^2)$. Numerical experiments completely confirm theoretic results. Therefore, this method presented in this paper is of practical importance in scientific computation.

AMS subject classifications: 35Q10, 65N30, 76D05

Key words: Incompressible flow, stabilized finite volume method, inf-sup condition, local Gauss integral, two-scale method.

1 Introduction

Finite volume method (FVM) is an important numerical tool for solving partial differential equations. It has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer, and petroleum engineering. The FVM is intuitive in that it is directly based on local conservation of mass, momentum, or energy over volumes (control volumes or co-volumes). It lies somewhere between the finite element methods (FEM) and the finite difference methods (FDM) and has the flexibility similar to that of the

*Corresponding author.
Email: jianhongy_1977@126.com (J. H. Yang)

FEM for handling complicated geometries. Implementation is comparable to that of the FDM. The FVM is also referred to as the control volume method, the covolum method, or the first order generalized difference method [4, 6, 8–10, 37]. Its theoretical analysis is much more complex than that of the FEM.

In the paper [18], the authors consider the finite volume algorithm for solving the S-tokes problems and shows that it has optimal efficiency for the velocity in L^2-norm in the sense that if $f\in[H^1(\Omega)]^d$, $d=2,3$. A superconvergence result is established for the stationary Navier-Stokes equations by a stabilized finite volume method and L^2-projection on a coarse mesh [16, 30, 31]. Then a new stabilized FVM is studied and developed by Li et al. in [30] for the stationary incompressible flow. This method is based on a local Gauss integration technique and uses the lowest equal order finite element pair P_1-P_1 that do not satisfy the inf-sup (LBB) stability conditions [13, 22, 24, 25]. Stability and convergence of the optimal order in the L^2-norm and H^1-norm for velocity and the L^2-norm for pressure are obtained. A new duality for the incompressible flow is introduced to establish the convergence of the optimal order in the L^2-norm for velocity [19, 30].

Two-scale schemes have been applied to a variety of the steady semilinear equation-s by Xu [35, 36], the steady non-linear saddle point problems with the non-linear constraints by Niemisto in his thesis [33], and the steady incompressible flow by Layton, Li and Hou [17,20,23,26–29], and later on by Girault and Lions with particular emphasis on the three-dimensional problem on domains with corners [11].At the same time, Chen and Liu [7] have also studied this method for semilinear parabolic problems. However, more study is required for the stationary incompressible flow of finite volume approximation. Moreover, the theoretic analysis of two-scale stabilized FVM is more difficult than that of FEM.

In this article, we combine stabilized FVM based on P_1-P_1 element with two-scale strategy to obtain a two-scale stabilized FVM for the two-dimensional incompressible flow.The main procedure is stated as follows:

Step 1 Solve a small non-linear system on the coarse scale.

Step 2 Solve a linear system on the fine scale.

The convergence of the optimal order in the H^1-norm for velocity and L^2-norm for pressure are obtained. We choose the two-scale spaces as two conforming finite element spaces V_H, Q_H and V_h, Q_h on one coarse grid with mesh size H and one fine grid with mesh size $h\ll H$. The two-scale method consist of solving a small incompressible flow problem on the coarse mesh and then solving a linear Stokes problem on the fine mesh. Then, we prove that the two-scale stabilized finite volume solution $(u^h,\ p^h)$ has the following error estimate:

$$\|\nabla(u-u^h)\|_0+\|p-p^h\|_0\le C(h+H^2).$$

In solving approximate solution of the stationary incompressible flow on a fine mesh sizes satisfying $h=\mathcal{O}(H^2)$, the error analysis shows that the two-scale stabilized FVM provides the same approximate solution with the convergence rate of the same order as the usual stabilized finite volume method. Finally, numerical tests here showed the stability and efficiency of the presented method.

The rest of the paper is organized as follows: In the next section, we introduce some notations of the stationary incompressible flow and the stabilized FEM. Then, in the third section, the stabilized FVM is defined, and some useful theorems are shown. A two-scale stabilized FVM is constructed and optimal order estimates for this method are obtained, in the fourth section. Finally, numerical results to check the theoretical results are provided in the fifth section.

2 A stabilized finite element method

let Ω be a bounded domain in $\mathbf{R}^2$, assumed to have a Lipschitz-continuous boundary $\partial\Omega$ and to satisfy a further condition stated in **(A1)** below. The equations of stationary incompressible flow are considered as follows:

$$-\nu\Delta u+\nabla p+(u\cdot\nabla)u=f \qquad \text{in } \Omega, \tag{2.1a}$$

$$\nabla\cdot u=0 \qquad \text{in } \Omega, \tag{2.1b}$$

$$u|_{\partial\Omega}=0, \tag{2.1c}$$

where the symbols Δ, ∇, and $\nabla\cdot$ denote the Laplace, gradient, and divergence operators, respectively, and $u=(u_1,u_2)$ is the velocity vector, $p=p(x,y)$ is the pressure, $f=f(x,y)$ the prescribed body force, and $\nu>0$ the viscosity of the fluid.

We set

$$V\equiv H_0^1(\Omega)^2, \quad Q\equiv L_0^2(\Omega), \quad X\equiv L^2(\Omega)^2, \quad D(A)\equiv H^2(\Omega)^2\cap V,$$

where

$$L_0^2(\Omega)\equiv\Big\{q\in L^2(\Omega);\ \int_\Omega q dx=0\Big\}.$$

Here $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^2$ for $i=0,1,2$. we denotes by $(\cdot,\cdot)$ and $(|\cdot|)$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$ respectively. The space $H_0^1(\Omega)$ and V are equipped with their usual scalar product and norm

$$((u,v))=(\nabla u,\nabla v), \quad \|u\|_1=((u,u))^{1/2}.$$

It is well known that for each $v\in V$ hold the following inequalities:

$$\|v\|_{L^4}\le C_0\|v\|_0^{1/2}\|\nabla v\|_0^{1/2}, \qquad \|v\|_0\le C_1\|\nabla v\|_0, \tag{2.2a}$$

$$\|\nabla v\|_{L^4}\le C_0\|\nabla v\|_0^{1/2}\|v\|_2^{1/2}, \qquad \|v\|_\infty\le C_2\|v\|_0^{1/2}\|v\|_2^{1/2}, \tag{2.2b}$$

the generic positive constant C (with or without a subscript) depends only on Ω. Subsequently, C will denote a generic positive constant depending at most on the data Ω,v and f. As mention above, a further assumption on Ω is presented:

(A1) Assume that Ω regular so that the unique solution $(v,q)\in(V,Q)$ of the steady Stokes problem

$$-\Delta v+\nabla q=g \quad \text{in } \Omega, \quad \nabla\cdot v=0 \quad \text{in } \Omega, \quad v|_{\partial\Omega}=0, \tag{2.3}$$

for a prescribed $g\in X$ exists and satisfies

$$\|v\|_2+\|q\|_1\le C\|g\|_0. \tag{2.4}$$

Then the mixed variational form of (2.1a)-(2.1c) is to seek $(u,p)\in(V,Q)$, such that $\forall(v,q)\in(V,Q)$ hold equations

$$a(u,v)-d(v,p)+d(u,q)+b(u,u,v)=(f,v), \tag{2.5}$$

where the continuous bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and the trilinear term $b(\cdot,\cdot,\cdot)$ are defined by

$$a(u,v)=\nu((u,v))=\nu(\nabla u,\nabla v), \qquad \forall u,v\in V, \tag{2.6a}$$

$$d(v,q)=(\nabla\cdot v,q), \qquad \forall v\in V, \quad \forall q\in Q, \tag{2.6b}$$

$$\begin{aligned} b(u,v,w)&=((u\cdot\nabla)v,w)+\frac{1}{2}((\mathrm{div}u)v,w)\\ &=\frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v), \qquad \forall u,v,w\in V, \end{aligned} \tag{2.6c}$$

and the trilinear term satisfies

$$b(u,v,w)=-b(u,w,v), \tag{2.7a}$$

$$|b(u,v,w)|\le C_3\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0, \quad \forall u,v,w\in V, \tag{2.7b}$$

$$|b(u,v,w)|+|b(v,u,w)|+|b(w,u,v)|\le C_3\|u\|_2\|v\|_1\|w\|_0, \tag{2.7c}$$

where $u\in V$, $v\in D(A)$, $w\in X$.

Moreover, a generalized bilinear forms on $(V,Q)\times(V,Q)$ by

$$\mathcal{B}((u,p),(v,q))=a(u,v)-d(v,p)+d(u,q), \tag{2.8}$$

so (2.5) can be rewrite in a compact form: find $(u,p)\in(V,Q)$ such that

$$\mathcal{B}((u,p),(v,q))+b(u,u,v)=(f,v). \tag{2.9}$$

The existence and uniqueness solution of (2.9) can be found in [12,34].

Theorem 2.1 (see [30]). *If $\nu>0$ and $f\in X$ satisfy*

$$1-\frac{C_1C_3}{\nu^2}\|f\|_0>0, \tag{2.10}$$

then the variational problem (2.9) *admits a unique solution* $(u,p)\in(D(A),H^1(\Omega)\cap Q)$ *such that*

$$\|u\|_1\le\frac{C_1}{\nu}\|f\|_0,\quad \|u\|_2+\|p\|_1\le C\|f\|_0, \tag{2.11}$$

where the positive constants C_1 and C_3 are given by (2.2a) *and* (2.7b).

In general, (2.9) has more than one solution. Uniqueness is guaranteed if the viscosity and body force satisfy (2.10). For $h>0$, let K_h be a triangulation of Ω into triangles, assumed to be shape regular in the usual sense [3, 5, 12]. Associated with K_h we introduce finite dimensional subspaces $(V_h,Q_h)\in(V,Q)$. For these spaces we assume that the following approximation properties hold: For $(v,q)\in(D(A),H^1(\Omega)\cap Q)$, there exist approximations $I_hv\in V_h$ and $J_hq\in Q_h$ such that

$$\|v-I_hv\|_0+h\|v-I_hv\|_1\le Ch^2\|v\|_2, \tag{2.12a}$$

$$\|q-J_hq\|_0+h\|q-J_hq\|_1\le Ch\|q\|_1. \tag{2.12b}$$

In particular, the interpolation operator I_h satisfies

$$\|I_hv\|_1\le C\|v\|_1. \tag{2.13}$$

Due to the quasi-uniformness of the triangulation K_h, the following properties hold [5, 34]:

$$\|v_h\|_1\le C_4h^{-1}\|v_h\|_0,\quad \|v_h\|_\infty\le C_5|logh|^{1/2}\|v_h\|_1,\quad \forall v_h\in X_h. \tag{2.14}$$

We define a discrete analogue A_h [14]of the Laplace operator Δ by

$$(A_hu_h,v_h)=(\nabla u_h,\nabla v_h),\quad u_h,v_h\in V_h,$$

and define

$$\bar{V}_h=\{v_h\in V_h:d(v_h,q_h)=0,\ \forall q_h\in Q_h\}.$$

The restriction of A_h to $\bar{V}_h$ is invertible, with the inverse A_h^{-1}. we define the discrete Sobolev norm of $A_h^{1/2}$ on $\bar{V}_h$ by

$$\|v_h\|_{1,h}=\|A_h^{1/2}v_h\|_0,\quad v_h\in\bar{V}_h.$$

In this paper, we use the lowest equal-order pairs (V_h,Q_h) to approximate problem (2.1a)-(2.1c),

$$V_h=\{v_h\in(C^0(\Omega))^2\cap V:v_h|_K\in(P_1(K))^2,\ \forall K\in K_h\},$$
$$Q_h=\{q_h\in C^0(\Omega)\cap Q:q_h|_K\in P_1(K),\ \forall K\in K_h\},$$

where $P_1(K)$ is the set of linear functions on element K. But it does not satisfy the discrete inf-sup condition:

$$\sup_{0\neq v_h\in V_h}\frac{d(v_h,q_h)}{\|v_h\|_1}\geq\beta_1\|q_h\|_0,\quad q_h\in Q_h, \tag{2.15}$$

where the constant $\beta_1>0$ is independent of h. The local Gauss integration term is used to fulfill this condition [30]:

$$G_h(p_h,q_h)=\Sigma_{K\in K_h}\Big\{\int_{K,2}p_hq_hdx-\int_{K,1}p_hq_hdx\Big\},\quad p_h,q_h\in Q_h, \tag{2.16}$$

where $\int_{K,i}g(x)dx$ indicates an appropriate Gauss integral over K that is exact for polynomials of degree i, $i=1,2$ and $g(x)=p_hq_h$ is a polynomial of degree not greater than two. In particular, the trial function $p_h\in Q_h$ must be projected to piecewise constant space W_h defined below when $i=1$ for any $q_h\in Q_h$. Consequently, we define the L^2-projection operator $\Pi_h:L^2(\Omega)\rightarrow W_h$:

$$(p,q_h)=(\Pi_hp,q_h),\quad \forall p\in L^2(\Omega),\quad q_h\in W_h, \tag{2.17}$$

where $W_h\subseteq L^2(\Omega)$ denotes the piecewise constant space associated with the triangulation K_h. The following properties of the projection operator Π_h can be proved [5]

$$\|\Pi_hp\|_0\leq C\|p\|_0,\qquad \forall p\in L^2(\Omega), \tag{2.18a}$$

$$\|p-\Pi_hp\|_0\leq Ch\|p\|_1,\qquad \forall p\in H^1(\Omega). \tag{2.18b}$$

As a result of (2.17), the bilinear form $G_h(\cdot,\cdot)$ can be expressed as

$$\begin{aligned}G_h(p_h,q_h)&=(p_h-\Pi_hp_h,q_h)\\&=(p_h-\Pi_hp_h,q_h-\Pi_hq_h),\quad p_h,q_h\in Q_h.\end{aligned} \tag{2.19}$$

So the corresponding discrete variational formulation of (2.9) for the incompressible flow is recast: to find $(\bar{u}_h,\bar{p}_h)\in(V_h,Q_h)$ such that

$$\mathcal{B}_h((\bar{u}_h,\bar{p}_h),(v_h,q_h))+b(\bar{u}_h,\bar{u}_h,v_h)=(f,v_h),\quad \forall(v_h,q_h)\in(V_h,Q_h), \tag{2.20}$$

where the bilinear form $\mathcal{B}_h(\cdot,\cdot)$ on $(V_h,Q_h)\times(V_h,Q_h)$ is given by

$$\begin{aligned}&\mathcal{B}_h((\bar{u}_h,\bar{p}_h),(v_h,q_h))\\&=a(\bar{u}_h,v_h)-d(v_h,\bar{p}_h)+d(\bar{u}_h,q_h)+G_h(\bar{p}_h,q_h),\quad (\bar{u}_h,\bar{p}_h),(v_h,q_h)\in(V_h,Q_h).\end{aligned} \tag{2.21}$$

The following theorem establishes the continuity and weak coercivity of (2.21) for the equal-order finite element pair P_1-P_1 [1,2,22].

Theorem 2.2 (see [22]). *Let (V_h,Q_h) be defined as above, then $\mathcal{B}_h((\cdot,\cdot),(\cdot,\cdot))$ in* (2.21) *hold*

$$|\mathcal{B}_h((\bar{u}_h,\bar{p}_h),(v_h,q_h))|\le C(\|\nabla\bar{u}_h\|_0+\|\bar{p}_h\|_0)(\|\nabla v_h\|_0+\|q_h\|_0), \tag{2.22}$$

and

$$\beta_2(\|\nabla\bar{u}_h\|_0+\|\bar{p}_h\|_0)\le \sup_{(v_h,q_h)\in(V_h,Q_h)}\frac{|\mathcal{B}_h((\bar{u}_h,\bar{p}_h),(v_h,q_h))|}{\|\nabla v_h\|_0+\|q_h\|_0}, \tag{2.23}$$

where $\beta_2>0$ is independent of h.

By Theorem 2.2, Eq. (2.20) can be shown to have a unique solution [15]. Moreover, the optimal error estimate for the finite element solution $(\bar{u}_h,\bar{p}_h)$ holds for sufficiently small $h>0$ [13]:

$$\|u-\bar{u}_h\|_0+h(\|\nabla(u-\bar{u}_h)\|_0+\|p-\bar{p}_h\|_0)\le Ch^2(\|u\|_2+\|p\|_1+\|f\|_0). \tag{2.24}$$

3 A stabilized finite volume method

In order to construct of the FVM scheme for the problem (2.1a)-(2.1c), we will introduce a dual partition $\tilde{K}_h$ based on the partition K_h whose element are called control volumes. Let N_h be the set containing all the interior nodes associated with the triangulation K_h, and N be the total number of the nodes. The dual mesh can be constructed by the following rule: For each element $K\in K_h$ with vertices P_j, $j=1,2,3$, select its barycenter Q_j and the midpoint M_j on each of the edges of K, and construct the control volumes in $\tilde{K}_h$ by connecting Q_j to M_j as shown in Fig. 1.

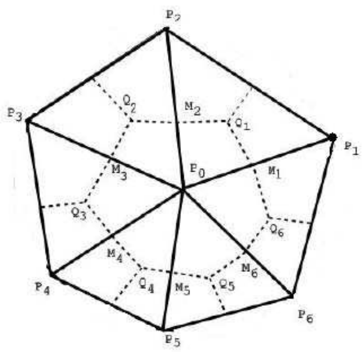

Figure 1: Control volumes associated with triangles.

Associated with $\tilde{K}_h$, the dual finite element space is defined as

$$\tilde{V}_h=\left\{\tilde{v}\in\left(L^2(\Omega)\right)^2:\tilde{v}|_{\tilde{K}}\in P_0(\tilde{K}),\ \forall\tilde{K}\in\tilde{K}_h;\ \tilde{v}|_{\partial\tilde{K}}=0\right\}.$$

Obviously, the dimensions of $\tilde{V}_h$ same as V_h. We defined an invertible linear mapping $\Gamma_h: V_h \to \tilde{V}_h$ such that for

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\varphi_j(x), \quad x \in \Omega, \quad v_h \in V_h, \tag{3.1}$$

we have

$$\Gamma_h v_h(x) \equiv v_h^* = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \quad x \in \Omega, \quad v_h \in V_h, \tag{3.2}$$

where $\{\varphi_j\}$ indicates the basis of the finite element space V_h and $\{\chi_j\}$ denotes the basis of the finite volume space $\tilde{V}_h$ that are the characteristic functions associated with the dual partition $\tilde{K}_h$:

$$\chi_j(x) = \begin{cases} 1, & \text{if } x \in \tilde{K}_j \in \tilde{K}_h, \\ 0, & \text{otherwise.} \end{cases}$$

The above idea of connecting the trial and test spaces in the Petrov-Galerkin method through the mapping Γ_h was first introduced in [32] the context of elliptic problems.

Theorem 3.1 (see [30]). *Let $K \in K_h$. If $v_h \in V_h$ and $1 \le r \le \infty$, then*

$$\int_K (v_h - v_h^*)dx = 0, \tag{3.3a}$$
$$\|v_h - v_h^*\|_{0,r,K} \le C_6 h_K \|v_h\|_{1,r,K}, \tag{3.3b}$$
$$\|v_h^*\|_0 \le C_7 \|v_h\|_0, \tag{3.3c}$$

where h_K is the diameter of the element K.

Here, a useful map Γ_h is introduced to build relationship between FEM and FVM. To introduce a variational formulation of the FVM, we multiply Eq. (2.1a) by $v_h^* \in \tilde{V}_h$ and integrate over the dual elements $\tilde{K} \in \tilde{K}_h$, multiply Eq. (2.1b) by $q_h \in Q_h$ and integrate over the primal elements $K \in K_h$, and apply Green's formula to obtain the following bilinear forms:

$$A(u_h, v_h^*) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} \frac{\partial u_h}{\partial \vec{n}} ds, \qquad u_h, v_h \in V_h,$$

$$D(v_h^*, p_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} p_h \vec{n} ds, \qquad p_h \in Q_h,$$

$$(f, v_h^*) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f dx, \qquad v_h \in V_h,$$

where $\vec{n}$ is the unit normal outward to $\partial\tilde{K}_j$. Also, we define the trilinear form $b(\cdot,\cdot,\cdot)$: $V_h \times V_h \times \tilde{V}_h \rightarrow \mathbf{R}$ for the FVM

$$b(u_h,v_h,w_h^*) = ((u_h \cdot \nabla)v_h,w_h^*) + \frac{1}{2}((\mathrm{div}u_h)v_h,w_h^*), \quad \forall u_h,v_h,w_h \in V_h. \tag{3.4}$$

Then the stabilized FVM for the incompressible flow (2.1a)-(2.1c) is to find $(u_h,p_h) \in (V_h,Q_h)$ such that

$$\mathcal{C}_h((u_h,p_h),(v_h,q_h)) + b(u_h,u_h,v_h^*) = (f,v_h^*), \quad \forall (v_h,q_h) \in (V_h,Q_h), \tag{3.5}$$

where the bilinear term $\mathcal{C}_h(\cdot,\cdot)$ on $(V_h,Q_h) \times (V_h,Q_h)$ is

$$\mathcal{C}_h((u_h,p_h),(v_h,q_h)) = A(u_h,v_h^*) + D(v_h^*,p_h) + d(u_h,q_h) + G_h(p_h,q_h), \tag{3.6}$$

such that [18,30]

$$|\mathcal{C}_h((u_h,p_h),(v_h,q_h))| \leq C(\|\nabla u_h\|_0 + \|p_h\|_0)(\|\nabla v_h\|_0 + \|q_h\|_0), \tag{3.7}$$

and

$$\begin{aligned} &\beta_3(\|\nabla u_h\|_0 + \|p_h\|_0) \\ \leq &\sup_{(v_h,q_h)\in(V_h,Q_h)} \frac{|\mathcal{C}_h(u_h,p_h),(v_h,q_h))|}{\|\nabla v_h\|_0 + \|q_h\|_0}, \quad \forall (u_h,p_h),(v_h,q_h) \in (V_h,Q_h), \end{aligned} \tag{3.8}$$

where $\beta_3 > 0$ is independent of h.

The optimal analysis of finite element approximation based on the relationship between the FEM and the FVM can be found in [18]. The equivalence is stated for the bilinear term $A(\cdot,\cdot)$ and $D(\cdot,\cdot)$ as follows:

Theorem 3.2 (see [18,37]). *There holds*

$$A(u_h,v_h^*) = a(u_h,v_h), \qquad \forall u_h,v_h \in V_h, \tag{3.9a}$$

$$D(v_h^*,q_h) = -d(v_h,q_h), \qquad \forall (v_h,q_h) \in (V_h,Q_h), \tag{3.9b}$$

with the following properties:

$$A(u_h,v_h^*) = A(v_h,u_h^*), \tag{3.10a}$$

$$|A(u_h,v_h^*)| \leq C\|\nabla u_h\|_0 \|\nabla v_h\|_0, \tag{3.10b}$$

$$|A(v_h,v_h^*)| \geq C\|v_h\|_1^2. \tag{3.10c}$$

The existence and the uniqueness of solution to the Eq. (3.5) are proved in Theorem 3.3.

Theorem 3.3 (see [30]). *For each $h>0$ such that*

$$0<h_0\leq\frac{1}{2}, \tag{3.11}$$

system (3.5) *admits a solution $(u_h,p_h)\in(V_h,Q_h)$. Moreover, if the viscosity $\nu>0$, the body force $f\in X$, and the mesh size $h>0$ satisfy*

$$0<h_0\leq\frac{1}{4},\quad 1-\frac{4C_1C_3C_7}{\nu^2}\|f\|_0>0, \tag{3.12}$$

then the solution $(u_h,p_h)\in(V_h,Q_h)$ is unique. Furthermore, it satisfies

$$\|\nabla u_h\|_0\leq\frac{2C_1C_7}{\nu}\|f\|_0, \tag{3.13a}$$

$$\|p_h\|_0\leq 2\beta_2^{-1}C_1C_7\|f\|_0\Big(1+\frac{2C_1C_3C_7}{\nu^2}\|f\|_0\Big), \tag{3.13b}$$

$$\|A_hu_h\|_0\leq\frac{2C_7}{\nu}\|f\|_0\Big(1+\frac{2^5C_1^4C_2^2C_7^4}{\nu^4}\|f\|_0^2\Big). \tag{3.13c}$$

Moreover, the estimates for velocity in the H^1- and L^2-norm and the pressure in the L^2-norm is presented as follows.

Theorem 3.4 (see [30]). *Assume that $h>0$ satisfies* (3.11) *and $f\in X$ and $\nu>0$. Let $(u,p)\in(V,Q)$ and $(u_h,p_h)\in(V_h,Q_h)$ be the solution of* (2.9) *and* (3.5), *respectively. Then it holds*

$$\|\nabla(u-u_h)\|_0+\|p-p_h\|_0\leq Ch(\|u\|_2+\|p\|_1+\|f\|_0), \tag{3.14}$$

and

$$\|u-u_h\|_0\leq Ch^2(\|u\|_2+\|p\|_1+\|f\|_1). \tag{3.15}$$

4 Two-scale picard stabilized finite volume method

In this section, we aim to present the two-scale stabilized FVM and derive some optimal bounds of the errors.

let H and $h\ll H$ will be two real positive parameters tending to 0. A coarse mesh triangulation of $K_H(\Omega)$ of Ω is made and a fine mesh triangulation $K_h(\Omega)$ is generated by a mesh refinement process to $K_H(\Omega)$. The conforming finite element space pairs (V_h,Q_h) and $(V_H,Q_H)\subset(V_h,Q_h)$ based on the triangulations $K_h(\Omega)$ and $K_H(\Omega)$ respectively, are constructed as like in Section 2. Two-scale stabilized finite volume approximation is defined as follows.

Step1 Solve the stabilized Navier-Stokes problem on a coarse mesh: find $(u_H,p_H)\in(V_H,Q_H)$ such that for all $(v_H,q_H)\in(V_H,Q_H)$

$$\mathcal{C}_h((u_H,p_H),(v_H,q_H))+b(u_H,u_H,v_H^*)=(f,v_H^*). \tag{4.1}$$

Step2 Solve the stabilized Stokes problem on a fine mesh: find $(u^h,p^h)\in(V_h,Q_h)$ such that for all $(v_h,q_h)\in(V_h,Q_h)$

$$\mathcal{C}_h((u^h,p^h),(v_h,q_h))+b(u_H,u^h,v_h^*)=(f,v_h^*). \tag{4.2}$$

To derive error estimates for the finite volume solution (u^h,p^h), we define a projection operator $(R_h,T_h):(V,Q)\to(V_h,Q_h)$ by

$$\begin{aligned}&\mathcal{C}_h((u-R_h(u,p),p-T_h(u,p)),(v_h,q_h))\\ =&G_h(p,q_h),\quad \forall(u,p)\in(V,Q),\ (v_h,q_h)\in(V_h,Q_h),\end{aligned} \tag{4.3}$$

which satisfies the following stability and approximation properties:

Theorem 4.1 (see [19])**.** *Under the assumption of* **(**A1**)***, the projection operator* (R_h,T_h) *satisfies*

$$\|R_h(u,p)\|_1+\|T_h(u,p)\|_0\le C(\|u\|_1+\|p\|_0), \tag{4.4}$$

and

$$\|\nabla(u-R_h(u,p))\|_0+\|p-T_h(u,p)\|_0\le Ch(\|u\|_2+\|p\|_1), \tag{4.5}$$

for all $(u,p)\in(D(A),H^1(\Omega)\cap Q)$.

For convenience, we set $e=R_h(u,p)-u^h$, $\eta=T_h(u,p)-p^h$. Then a error estimates for the two-scale error $u-u^h$ and $p-p^h$ is derived as follows.

Theorem 4.2. *Under the assumptions of Theorems 2.1 and Theorems 3.1-3.3, the following two-scale stabilized finite volume solution* (u^h,p^h) *satisfy the following error estimate*

$$\|\nabla(u-u^h)\|_0+\|p-p^h\|_0\le C(h+H^2). \tag{4.6}$$

Proof. Using (4.3), Theorem 3.2 and subtracting (4.2) from (2.9) with $(v_h,q_h)=(e,\eta)$, we have

$$\mathcal{C}_h((e,\eta),(e,\eta))+b(u,u,e)-b(u_H,u^h,e^*)=(f,e-e^*). \tag{4.7}$$

Obviously, it follows that

$$\mathcal{C}_h((e,\eta),(e,\eta))\ge\nu\|\nabla e\|_0^2, \tag{4.8a}$$
$$|(f,e-e^*)|\le\|f\|_0\|e-e^*\|_0\le Ch\|f\|_0\|\nabla e\|_0. \tag{4.8b}$$

For the trilinear terms, it is easy to find that

$$\begin{aligned}|b(u,u,e)-b(u_H,u^h,e^*)|=&|b(u,u,e)-b(u_H,u,e)+b(u_H,u,e)-b(u_H,u^h,e)\\ &+b(u_H,u^h,e)-b(u_H,u^h,e^*)|\\ =&|b(u-u_H,u,e)+b(u_H,u-u^h,e)+b(u_H,u^h,e-e^*)|\\ \le&|b(u-u_H,u,e)|+|b(u_H,u-u^h,e)|+|b(u_H,u^h,e-e^*)|.\end{aligned} \tag{4.9}$$

By (2.7c) and (3.15)

$$|b(u-u_H,u,e)|\le C\|u\|_2\|u-u_H\|_0\|\nabla e\|_0\le CH^2\|\nabla e\|_0. \tag{4.10}$$

Using (2.7a), (2.7b), (4.5) and **(A1)** gives that

$$\begin{aligned}|b(u_H,u-u^h,e)|&=|b(u_H,u-R_h(u,p)+R_h(u,p)-u^h,e)|\\&=|b(u_H,u-R_h(u,p),e)+b(u_H,e,e)|\\&=|b(u_H,u-R_h(u,p),e)|\\&\le C\|\nabla u_H\|_0\|\nabla(u-R_h(u,p))\|_0\|\nabla e\|_0\\&\le Ch(\|u\|_2+\|p\|_1)\|\nabla e\|_0\\&\le Ch\|\nabla e\|_0.\end{aligned} \tag{4.11}$$

As for the third term on the right of (4.9), a simple calculation is used to obtain by setting $E=R_h(u,p)-u$

$$\begin{aligned}|b(u_H,u^h,e-e^*)|&=|b(u_H,u^h-u,e-e^*)+b(u_H,u,e-e^*)|\\&=|b(u_H,-e+R_h(u,p)-u,e-e^*)+b(u_H,u,e-e^*)|\\&=|b(u_H-u,e,e-e^*)+b(u,e,e-e^*)+b(u_H,E,e-e^*)+b(u_H,u,e-e^*)|\\&\le|b(u_H-u,e,e-e^*)|+|b(u,e,e-e^*)|+|b(u_H,E,e-e^*)|+|b(u_H,u,e-e^*)|\\&=I_1+I_2+I_3+I_4.\end{aligned} \tag{4.12}$$

Using the Hölder inequality, (2.2a) and Theorem 3.4 gives

$$\begin{aligned}I_1&=|b(u_H-u,e,e-e^*)|\\&\le(\|u_H-u\|_{L^4}\|\nabla e\|_0+\|e\|_{L^4}\|\nabla(u_H-u)\|_0)\|e^*-e\|_{L^4}\\&\le Ch(\|u_H-u\|_0^{1/2}\|\nabla(u_H-u)\|_0^{1/2}\|\nabla e\|_0\\&\qquad+\|e\|_0^{1/2}\|\nabla e\|_0^{1/2}\|\nabla(u_H-u)\|_0)\|\nabla e\|_{L^4}\\&\le C(H^{5/2}+H^2)\|\nabla e\|_0^2.\end{aligned} \tag{4.13}$$

Due to (2.7c) and (3.3b)

$$I_2=|b(u,e,e-e^*)|\le C\|u\|_2\|\nabla e\|_0\|e-e^*\|_0\le Ch\|u\|_2\|\nabla e\|_0^2. \tag{4.14}$$

Noting that [17]

$$\|v\|_{L^6}\le C\|\nabla v\|_0,\qquad v\in V, \tag{4.15a}$$

$$\|v_h\|_{L^\infty}+\|\nabla v_h\|_{L^3}\le C\|\nabla v_h\|_0^{1/2}\|A_hv_h\|_0^{1/2},\qquad v_h\in V_h, \tag{4.15b}$$

and the Hölder inequality, we have

$$\begin{aligned}I_3&=|b(u_H,E,e-e^*)|\\&\le C(\|u_H\|_{L^\infty}\|\nabla E\|_0+\|\nabla u_H\|_{L^3}\|E\|_{L^6})\|e-e^*\|_0\\&\le Ch\|\nabla u_H\|_0^{1/2}\|A_Hu_H\|_0^{1/2}\|\nabla E\|_0\|\nabla e\|_0\\&\le Ch\|\nabla e\|_0.\end{aligned} \tag{4.16}$$

Similarly, by (2.7c),

$$\begin{aligned} I_4 =& |b(u_H,u,e-e^*)| \\ \leq& |b(u_H-u,u,e-e^*)| + |b(u,u,e-e^*)| \\ \leq& (\|\nabla(u_H-u)\|_0\|u\|_2 + \|u\|_2\|u\|_1)\|e-e^*\|_0 \\ \leq& C(h+H^2)\|\nabla e\|_0. \end{aligned} \tag{4.17}$$

Combining all these inequality with (4.7), gives

$$[\nu - C(h+H^{5/2})]\|\nabla e\|_0^2 \leq C(h+H^2)\|\nabla e\|_0. \tag{4.18}$$

If the coarse mesh scale H is sufficiently small, such that

$$[1-C\nu^{-1}(h+H^{5/2})] > \frac{1}{2},$$

it holds

$$\frac{\nu}{2}\|\nabla e\|_0^2 \leq C(h+H^2)\|\nabla e\|_0. \tag{4.19}$$

Then, we have

$$\|\nabla e\|_0 \leq C(h+H^2). \tag{4.20}$$

It follows from (3.8) that

$$\|\eta\|_0 \leq C(h+H^2). \tag{4.21}$$

Finally, using the triangle inequality, (4.5), (4.20), (4.21) and **(A1)**, we obtain

$$\begin{aligned} \|u-u^h\|_1 =& \|R^h(u,p)-u^h+u-R^h(u,p)\|_1 \\ \leq& \|R^h(u,p)-u^h\|_1 + \|u-R^h(u,p)\|_1 \\ =& \|\nabla e\|_0 + \|u-R^h(u,p)\|_1 \\ \leq& C(h+H^2), \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \|p-p^h\|_0 \leq& \|T^h(u,p)-p^h\|_0 + \|p-T^h(u,p)\|_0 \\ =& \|\eta\|_0 + \|p-T^h(u,p)\|_0 \\ \leq& C(h+H^2), \end{aligned} \tag{4.23}$$

which completes the proof of (4.6). ☐

5 Numerical experiments

In this section, in order to conform theoretical analysis of Theorem 4.2, we solve the two dimensional stationary incompressible flow by two-scale stabilized FVM approximated by the lowest equal-order finite element pairs.

Example 5.1 (Exact solution problem)**.** We consider problem (2.1a)-(2.1c) in the unit square $\Omega:\{0\le x,y\le 1\}$ with the right hand side function $f(x,y)$ generated by the exact solution

$$u(x,y)=(u_1(x,y),u_2(x,y)),\qquad p(x,y)=10(2x-1)(2y-1),$$
$$u_1(x,y)=10x^2(x-1)^2y(y-1)(2y-1),\qquad u_2(x,y)=-10x(x-1)(2x-1)y^2(y-1)^2,$$

with $\nu=1$.

Table 1 shows the H^1-error of the velocity and the L^2-error of the pressure. Detailedly, the first line and the second line that of the one-scale stabilized FEM and FVM, the third line presents the results of the two-scale FVM with scaling $h=\mathcal{O}(H^2)$, respectively. In

Table 1: Comparison of the one-scale method with the two-scale methods: $\nu=1$.

$1/H$	$1/h$	$CPU(s)$	$\frac{\\|\nabla(u-u^h)\\|_0}{\\|\nabla u\\|_0}$	$\frac{\\|p-p^h\\|_0}{\\|p\\|_0}$	u_{H^1}rate	p_{L^2}rate
2	4	0.032	1.53629	0.388664		
	4	0.109	1.54003	0.388749		
	4	0.047	1.53594	0.388679		
3	9	0.172	0.53869	0.0978643	1.29232	1.70068
	9	0.172	0.53944	0.0978444	1.29353	1.70120
	9	0.157	0.538738	0.0978795	1.29193	1.70054
4	16	0.657	0.249861	0.035939	1.33523	1.74107
	16	0.563	0.250038	0.0359277	1.33641	1.74128
	16	0.36	0.249876	0.0359448	1.33527	1.74108
5	25	0.641	0.142015	0.016532	1.26594	1.73999
	25	1.422	0.142068	0.0165269	1.26669	1.73995
	25	0.703	0.142026	0.0165345	1.26589	1.73999
6	36	2.938	0.0917077	0.0088001	1.19933	1.72916
	36	2.625	0.0917267	0.00879779	1.19978	1.72906
	36	1.406	0.0917165	0.00880141	1.19928	1.72919
7	49	5.609	0.0643537	0.005187	1.14892	1.71436
	49	5.344	0.0643616	0.00518621	1.14919	1.71422
	49	2.062	0.0643606	0.00518772	1.14887	1.71461
8	64	10.016	0.047807	0.00329814	1.11295	1.69575
	64	8.766	0.0478105	0.00329751	1.11312	1.69561
	64	4.313	0.0478123	0.00329773	1.11291	1.69645
9	81	13.844	0.0370032	0.00222498	1.08744	1.67091
	81	20.343	0.0370051	0.00222463	1.08755	1.67077
	81	6.89	0.0370077	0.00222393	1.08741	1.67239

addition, we want to compare the accuracy and relative efficiency of the one-scale FVM with the two-scale FVM. Thus, we compare the solution generated at a fixed value of h for the one-scale method with the results obtained using two-scale method where the finest grid has the same mesh spacing h. For example, for $h=1/49$ and $h=1/64$ in the one-scale method we can compare this with our results using two-scale method with $H=1/7$ and $H=1/8$. In two-scale computations when H was chosen so that $h=\mathcal{O}(H^2)$ for the two-scale FVM, we observed that there are same the optimal predicted rate of convergence between one-scale method and two-scale method, but the two-scale method was probably two times faster than the one-scale method in CPU times. Also, the finer grid calculations gave the greater savings in time and number of operations.

Example 5.2 (The driven cavity problem)**.** The driven cavity flow on a unit square with no-slip boundary conditions only in upper boundary with $u=(u_1,u_2)=(1,0)$. The cavity flows have been widely used as test cases for validating the incompressible fluid dynamics algorithm. In this example, the velocity field and the pressure level lines for $\nu=1$ are plotted in Fig. 2 by using the traditional one-scale stabilized FEM, FVM and the two-scale stabilized FVM with P_1-P_1 presented in this article, respectively. Obviously, these figures can show the stability but computational time that are respectively 1.172s, 1.206s, 0.641s for three different methods. Numerical results also validate the accuracy of the two-scale stabilized FVM for the stationary incompressible flow.

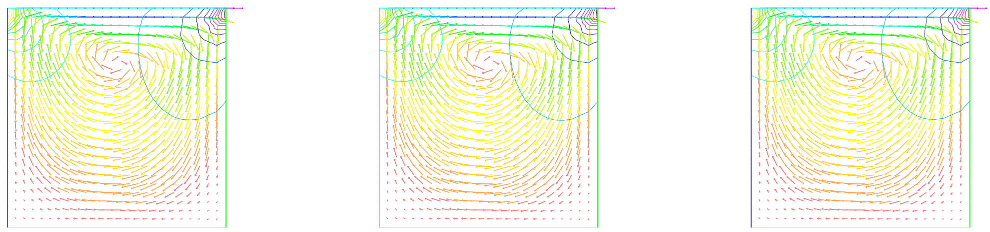

Figure 2: Comparison of the velocity field and the pressure level lines: $\nu=1$.

6 Conclusions

In this article, we combine stabilized FVM based on P_1-P_1 element with two-scale strategy to obtain a two-scale stabilized FVM for the two-dimensional Navier-Stokes equations. The two-scale method involves solving one small nonlinear Navier-Stokes problem on the coarse mesh and one linear Stokes problem on the fine mesh. We have obtained the optimal error estimates for the two-scale FVM. The error analysis shows that the two-scale stabilized FVM presented provides an approximate solution with the convergence rate of the same order as the usual stabilized finite volume solution solving the incompressible flow on a fine mesh for a related choice of mesh sizes satisfy $h=\mathcal{O}(H^2)$. Finally, two numerically tests are made to confirm that the two-scale method is efficient and saves a large amount of computational time compared to the one-scale method. Therefore, the two-scale FVM is suitable to solve some practical engineering problems arising in the fluid dynamics. Furthermore, we will apply two-scale FVM for the three-dimensional Navier-Stokes equations, and try to establish Multi-scale stabilized FVM [21] or other norm [20] for the stationary Navier-Stokes equations based on the idea of two-scale method in this paper.

Acknowledgements

This research was supported in part by the National Science Foundation of China (No. 11371031, NCET-11-1041).

References

[1] P. Bochev, C. R. Dohrmann and M. D. Gunzburger, *A computational study of stabilized, low order C^0 finite element approximations of Darcy equations*, Comput. Mech., 38 (2006), pp. 323–333.

[2] P. Bochev, C. R. Dohrmann and M. D. Gunzburger, *Stabilization of low-order mixed finite elements for the Stokes equations*, SIAM J. Numer. Anal., 44 (2006), pp. 82–101.

[3] Z. Chen, Finite Element Methods and Their Applications, Spring-Verlag, Heidelberg, 2005.

[4] Z. Chen, *The control volume finite element methods and their applications to multiphase flow*, Netw. Heterog. Media, 1 (2006), pp. 689–706.

[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[6] S. H. Chou and Q. Li, Error estimates in L^2, H^1 and L^∞ in co-volume methods for elliptic and parabolic problems: a unified approach, Math. Comput., 69 (2000), pp. 103–120.

[7] C. Chen and W. Liu, *Two-grid finite volume element methods for semilinear parabolic problems*, Appl. Numer. Math., 60 (2010), pp. 10–18.

[8] Z. Chen, R. Li and A. Zhou, *A note on the optimal L^2-estimate of finite volume element method*, Adv. Comp. Math., 16 (2002), pp. 291–303.

[9] Z. Cai, J. Mandel and S. Mc-ormick, *The finite volume element method for diffusion equations on general triangulations*, SIAM J. Numer. Anal., 28 (1991), pp. 392–403.

[10] R. E. Ewing, T. Lin and Y. Lin, *On the accuracy of the finite volume element method based on piecewise linear polynomials*, SIAM J. Numer. Anal., 39 (2002), pp. 1865–1888.

[11] V. Girault and J. L. Lions, *Two-grid finite-element schemes for the steady Navier-Stokes problem in polyhedra*, Portug. Math., 58 (2001), pp. 25–57.

[12] V. Girault and P. A. Raviart, Finite Element Method for Navier-Stokes Equations: Theory and Algorithms, Berlin, Heidelberg: Springer-Verlag, 1987.

[13] Y. He and J. Li, *A stabilized finite element method based on local polynomial pressure projection for the stationary Navier-Stokes equation*, Appl. Numer. Math., 58 (2008), pp. 1503–1514.

[14] J. G. Heywood and R. Rannacher, *Finite element approximations of nonstationary Navier-Stokes problem, Part I: Regularity of solutions and second-order spatial discretization*, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[15] Y. He, A. Wang and L. Mei, *Stabilized finite-element method for the stationary Navier-Stokes equations*, J. Eng. Math., 51 (2005), pp. 367–380.

[16] P. HUANG, T. ZHANG AND X. MA, *Superconvergence by L^2-projection for a stabilized finite volume method for the stationary Navier-Stokes equations*, Comput. Math. Appl., 62 (2011), pp. 4249–4257.

[17] W. LAYTON, *A two-level discretization method for the Navier-Stokes equations*, Comput. Math. Appl., 26 (1993), pp. 33–38.

[18] J. LI AND Z. CHEN, *A new stabilized finite volume method for stationary Stokes equations*, Adv. Comput. Math., 30 (2009), pp. 141–152.

[19] J. LI AND Z. CHEN, *On the semi-discrete stabilized finite volume method for the transient Navier-Stokes equations*, Adv. Comput. Math., 122 (2012), pp. 279–304.

[20] J. LI AND Z. CHEN, *Optimal L^2, H^1 and L^∞ analysis of finite volume methods for the stationary Navier-Stokes equations with large data*, Numerische Mathematik., 126 (2014), pp. 75–101.

[21] J. LI, Z. CHEN, AND Y. HE, *A stabilized multi-level method for non-singular finite volume solutions of the stationary 3D Navier-Stokes equations*, Numerische Mathematik., 122 (2012), pp. 279–304.

[22] J. LI AND Y. HE, *A new stabilized finite element method based on two local Gauss integration for the Stokes equations*, J. Comput. Appl. Math., 214 (2008), pp. 58–65.

[23] K. LI AND Y. HOU, *An AIM and one-step newton method for the Navier-Stokes equations*, Comput. Methods Appl. Mech. Eng., 190 (2001), pp. 6141–6155.

[24] J. LI, Y. HE AND Z. CHEN, *A new stabilized finite element method for the transient Navier-Stokes equations*, Comput. Meth. Appl. Mech. Eng., 197 (2007), pp. 22–35.

[25] J. LI, Y. HE AND Z. CHEN, *Performance of several stabilized finite element methods for the Stokes equations based on the lowest equal-order pairs*, Comput., 86 (2009), pp. 37–51.

[26] J. LI, Y. HE AND H. XU, *A multi-level stabilized finite element method for the stationary Navier-Stoke equations*, Comput. Meth. Appl. Mech. Eng., 196 (2007), pp. 2852–2862.

[27] W. LAYTON AND W. LENFERINK, *Two-level Picard-defect corrections for the Navier-Stokes equations at high Reynolds number*, Appl. Math. Comput., 69 (1995), pp. 263–274.

[28] W. LAYTON AND W. LENFERINK, *A multi-level mesh independence principle for the Navier-Stokes equations*, SIAM J. Numer. Anal., 33 (1996), pp. 17–30.

[29] W. LAYTON, H. K. LEE AND J. PETERSON, *Numerical solution of the stationary Navier-Stokes equations using a multilevel finite lement method*, SIAM J. Sci. Comput., 20 (1998), pp. 1–12.

[30] J. LI, L. SHEN AND Z. CHEN, *Convergence and stability of a stabilized finite volume method for stationary Navier-Stokes equations*, BIT Numer. Math., 50 (2010), pp. 823–842.

[31] J. LI, J. WU, Z. CHEN AND A. WANG, *Superconvergence of stabilized low order finite volume approximation for the three-dimensional stationary Navier-Stokes equations*, Int. J. Numer. Anal., Model, 9 (2012), pp. 419–431.

[32] R. LI AND P. ZHU, *Generalized difference methods for second order elliptic partial differential equations (I)-triangle grids*, Numer. Math J. Chinese Universities, 2 (1982), pp. 140–152.

[33] A. NIEMISTO, *FE-approximation of unconstrained optimal control like problems*, University of Jyvaskyla, Report, 70 (1995).

[34] R. TEMAM, Navier-Stokes Equations, Theory and Numerical Analysis, Amsterdam, North-Holland, 1983.

[35] J. XU, *A novel two-grid method for semilinear elliptic equations*, SIAM J. Sci. Comput., 15 (1994), pp. 231–237.

[36] J. XU, *Two-grid finite element discretization techniques for linear and nonlinear PDE*, SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777.

[37] X. YE, *On the relationship between finite volume and finite element methods applied to the Stokes equations*, Numer. Methods Partial Differ. Equ., 5 (2001), pp. 440–453.