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A Remark on the Courant-Friedrichs-Lewy Condition in Finite Difference Approach to PDE's

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Abstract. The Courant-Friedrichs-Lewy condition (The CFL condition) is appeared in the analysis of the finite difference method applied to linear hyperbolic partial differential equations. We give a remark on the CFL condition from a view point of stability, and we give some numerical experiments which show instability of numerical solutions even under the CFL condition. We give a mathematical model for rounding errors in order to explain the instability.

AMS subject classifications: 65M06, 65M12, 65G50

Key words: Numerical analysis, finite difference scheme, stability, CFL condition.

1 Introduction

The finite difference method (FDM) has been discussed as one of the mathematical tools to deal with partial differential equations before the era of digital computers till now, and R. Courant, K. O. Friedrichs and H. Lewy gave precise discussion about asymptotic behaviour of FDM solutions in [1]; we can see "we will find that for the case of the initial value problem for hyperbolic equations, convergence is obtained only if the ratio of the mesh widths in different directions satisfies certain inequalities which in turn depend on the position of the characteristics relative to the mesh" in [1] (this sentence is quoted from its English translation [2]). As is pointed out in [6], we note as follows; it is a necessary condition of convergence for FDM solution that the region of dependence of the finite

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difference schema contains that of the corresponding hyperbolic differential equation. We call this condition the Courant-Friedrichs-Lewy condition (CFL condition).

Let us introduce the CFL condition for the case

$$\frac{\partial}{\partial t}u(t,x)=c\frac{\partial}{\partial x}u(t,x), \qquad x\in\mathbb{R}, \quad t>0, \tag{1.1a}$$

$$u(0,x)=u_0(x), \qquad x\in\mathbb{R}, \tag{1.1b}$$

where c is a positive constant. Let Δt and Δx be increments along the t-direction and x-direction respectively, and we give a discretization by FDM

$$\frac{u_j^{k+1}-u_j^k}{\Delta t}=c\frac{u_{j+1}^k-u_j^k}{\Delta x}, \tag{1.2}$$

where u_j^k denotes the value corresponding to $u(k\Delta t,j\Delta x)$. Introduction of $\lambda:=\Delta t/\Delta x$ leads us to

$$u_j^{k+1}=c\lambda u_{j+1}^k+(1-c\lambda)u_j^k, \tag{1.3}$$

and the inequality $1-c\lambda\geq 0$ is the CFL condition for (1.2) and (1.3).

We sometimes encounter misunderstanding that the CFL condition is considered as a stability condition of the schema (1.2) or (1.3). The authors are afraid that it is caused by misunderstanding the proper meaning of the Lax equivalence theorem [5]. The analysis of stability of numerical solutions is one of the most important ones in the theory of FDM, and we may find its origin in the historic paper [7] of G. G. O'Brien et al. They gave, following the fundamental study of von Neumann, definition of stability so as to estimate numerical errors mainly coming from the rounding errors. Lax et al. [5] gave definition of stability as a discrete analogue to well-posedness of differential equations, and they declared that numerical errors were not taken into account in [5]. Lax et al., on the other hand, referred O'Brien et al. [7] as analysis of influence of numerical errors, and the authors are afraid that many researchers may have misunderstood the differences between their analyses.

Proper understanding of the analysis by O'Brien et al. [7] and by Lax et al. [5] is very much significant in order to understand real effects of rounding errors in computation by FDM. The authors consider it to be equivalent to give an answer to the question whether stability conditions guarantee stable computation less influenced by the rounding errors. For clear discussion, we need a mathematical model of propagation of the rounding errors in computation, but we should remark that modeling is dependent upon a system of floating point numbers on computers. When we restrict ourselves on computation of an evolutional finite difference schema, we notice that O'Brien et al. [7] adopted a naive model in their analysis of propagation of the rounding errors, and the authors should remark that it has close relation with stability analysis by Lax et al. [5]. Unfortunately it is inadequate in explanation of severe effects of the rounding errors occurred on *current* digital computers.

One of the aims of the present paper is to reconfirm the proper meaning of the CFL condition, and we have already mentioned it above. We remark again that it is a necessary condition of convergence of the finite difference solutions to the Cauchy problem of linear hyperbolic equations. It is not a sufficient condition for stability. The other of the aims is to show behaviours of the rounding errors for a *stable* finite difference schema through concrete computation. To this end, we shall give a mathematical model for propagation of the rounding errors arisen in the IEEE754 [3] computer environment in Section 2. We shall show some numerical examples to illustrate instability of numerical solutions of a stable schema in Section 3. They may not only diverge but may converge to zero, and both of them are quite different from the exact solutions of the difference schema. We note that they are unstable numerical computations of (1.2) or (1.3) for the case the CFL condition $1-c\lambda>0$ holds.

2 Mathematical modeling of the rounding errors

Influence of rounding errors is surely inevitable in computations which deal with real numbers on digital computers, but it is sometimes left outside of discussion in theoretical numerical analysis. Nonetheless we often encounter problems of accumulation of rounding errors in computation with digital computers and are obligated to come to know the difference between theories of computation and realistic computations. In the analysis of the rounding errors, we always focus on the fact that real numbers on digital computers have only finite significant figures, but we are often liable to forget the fact that basic arithmetics equipped on them admit also rounding errors.

Real numbers on computers are expressed as floating point numbers with finite figures of the binary codes, and each one, in general, contains rounding errors. We call, in the present paper, the four rules of the basic arithmetics on computers the computational arithmetics, and they are executed within some tolerance of rounding errors. We remark rounding errors appeared in real number computation are strictly ruled by the IEEE754 standard [3], but we also note that it leaves some flexibility or freedom to users. Hence the same real numbers may have different floating number expressions on different computers. As is well-known, the computational arithmetics do not satisfy the associative law.

In order to illustrate effects of rounding errors, let us consider addition $a+b$ for two real numbers a and b. Denoting computational real numbers with a bar $\bar{\cdot}$, we write $\overline{a}$ and $\overline{b}$ for the expression of a and b on computers. We also denote the computational addition by $\oplus$. Hence a mathematical statement $a+b$ is executed as $\overline{a}\oplus\overline{b}$ on computers, where all the rounding errors adherent to $\overline{a}\oplus\overline{b}$ are ruled by the IEEE754 standard now.

For the analysis of the rounding errors, we need a suitable mathematical model corresponding to their propagation in the processes of computation. Denoting the total sum of the rounding errors by δ, we know that

$$a+b+\delta \tag{2.1}$$

is a popular model of the rounding errors to $\overline{a}\oplus\overline{b}$, and it explains many types of computational phenomena. The authors strongly propose adopting, in place of (2.1),

$$(1+\epsilon)(a+b) \tag{2.2}$$

as a mathematical model corresponding to $\overline{a}\oplus\overline{b}$ in analysis of an evolutional finite difference schema, where $|\epsilon|$ is small and is determined by the rule of rounding-off. For a single calculation, there are no differences between (2.1) and (2.2), but they are completely different to each other in analysis of iterative calculations.

The mathematical modeling (2.2) comes from strict understanding of the rule of rounding errors of both real numbers and the computational arithmetics. We find analysis based on the models of the type of (2.2) in the famous book of Wilkinson [8]. The models of the type of (2.1) are convenient to know rough estimates of the rounding errors in finite numbers of calculations, and they often give reasonable estimates of accumulation of the rounding errors in polynomial order. For recurrent schema or evolutional finite difference schema, each computation should consist of finite number of calculations, but the numbers of calculation become huge as change of parameters in the schema; the total number of calculations for (1.2) or (1.3) becomes huge as a parameter Δt to zero. Hence the accumulation of the rounding errors sometimes does not only grow exponentially but also tends to zero for the cases. We do not mention here detailed estimates of these severe behaviours of the rounding errors, but we consider it possible to give their quantitative estimates by using the model of the type of (2.2). We remark that such estimates can be shown by the analysis not only of a finite difference scheme but also of orders of arithmetics appeared in the scheme, and they require delicate quantitative estimation of rounding errors for case by case.

Some other researchers have tried to discuss the same problem with us using the models of the type of (2.2), and we refer, for example, the paper of Jézéquel [4], which deals with the case of the heat equation. The authors should emphasize that we discuss the problem in connection with stability and convergence analysis of an evolutional finite difference schema. We shall show two types of instability of numerical solutions under the stability condition in the next section.

Returning to the historic work of O'Brien et al. [7], we notice that they adopted the models of the type (2.1). Applying their approach to (1.3), we obtain

$$\tilde{u}_j^{k+1}=c\lambda\tilde{u}_{j+1}^k+(1-c\lambda)\tilde{u}_j^k+\delta_j^k\Delta t, \tag{2.3}$$

where δ_j^k is the total sum of the rounding errors at k-th level. Since we can deal with the rounding errors as an inhomogeneous term of the difference equation (1.3), we are successful in Fourier analysis to conclude that $1-c\lambda\geq 0$ is a sufficient condition for stability in the sense of Lax. We should remark that rounding errors coming from the computational arithmetics are not taken into account in the formula (2.3) and that it is inadequate for explanation of influence of the rounding errors.

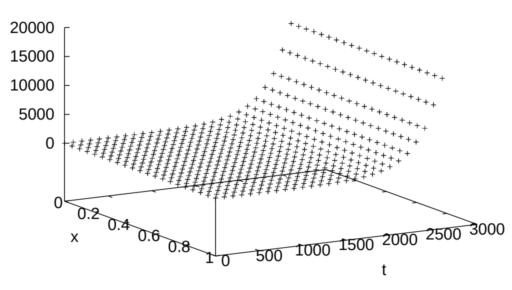
3 Numerical examples for instability

We show numerical examples for the equations (1.1a) and (1.1b) with $c=1$ and $u_0(x)=1$. Our finite difference schema is the formula (1.3) with a periodic boundary condition. The numerical computation is done for the case $\Delta x=1/20000$ and $\Delta t=1/30000$, and it is executed with the single precision of IEEE754. We note $\lambda=2/3$ in this case and that the stability condition in the sense of Lax holds. Furthermore we remark that $\lambda=2/3$ also satisfies the CFL condition.

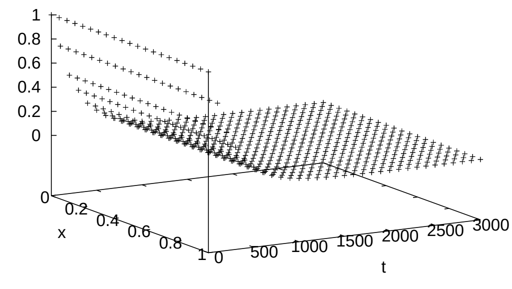
The authors are afraid that we are liable to imagine divergence of numerical solutions as their instability of numerical solutions, but we should notice that effects of the rounding errors are much complicated. The case (a) in Fig. 1 is the case that numerical solutions show divergence, but the case (b) shows extinction of numerical solutions.

Since the IEEE754 standard leaves users' flexibility in rounding-off within the single precision environment, we computed our finite difference schema in two ways; in the "round toward $+\infty$" mode and in the "round toward $-\infty$" mode. The former case corresponds to $\epsilon>0$ in (2.2), and the latter does to $\epsilon<0$. We remark again that numerical solutions may tend to zero as execution going and we should recognize that this case is one of the types of numerical instability. We easily notice instability of computation when numerical solutions show signs of divergence, but we are very much afraid that we may lose sign of instability for the latter case of extinction.

We conclude that numerical solutions may be unstable even under the stability condition in the sense of Lax, and the CFL condition is surely far from stability of numerical solutions. Without precise analysis of the behaviour of the rounding errors, we might misunderstand unstable numerical results as the aimed solution. The situation becomes worse for accurate computation that the mesh sizes are very small. The stability condition in the theory of FDM is that for the exact solutions to the difference equations, and they are different from numerical solutions obtained by computation using the difference equations. Furthermore, we should know that there are many types for numerical instability caused by the rounding errors.



(a) Within "Round toward $+\infty$" Mode



(b) Within "Round toward $-\infty$" Mode

Figure 1: Numerical Results with the Single Precision of IEEE754, $0\le t\le 3000$.

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