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Asymptotic Analysis of a Bingham Fluid in a Thin Domain with Fourier and Tresca Boundary Conditions

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Abstract. In this paper we prove first the existence and uniqueness results for the weak solution, to the stationary equations for Bingham fluid in a three dimensional bounded domain with Fourier and Tresca boundary condition; then we study the asymptotic analysis when one dimension of the fluid domain tend to zero. The strong convergence of the velocity is proved, a specific Reynolds limit equation and the limit of Tresca free boundary conditions are obtained.

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Key words: Free boundary problems, Bingham fluid, asymptotic approach, Tresca law, Reynolds equation.

1 Introduction

A Bingham fluid is a rigid viscoplastic fluid that is a particular kind of non-Newtonian fluid. The name is associated to Eugene C. Bingham (1878-1945) who, for the first time, in 1916, proposed a mathematical description for this visco-plastic behaviour [3]. There are many materials in nature and industry exhibiting the behavior of the Bingham medium. For example, heavy crude oils, colloid solutions, powder mixtures, metals under pressure treatment, blood in a capillary, foodstuffs and toothpaste. The mathematical models for such materials involve the constituent law for viscous incompressible fluids with an extra stress tensor component modeling the visco-plastic effects.

The analysis of the Bingham fluid flow variational inequality was carried out in [10], where the authors investigated existence, uniqueness and regularity of the solution for

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the steady and instationary flows in a reservoir. Existence and extra regularity results for the d-dimensional Bingham fluid flow problem with Dirichlet boundary conditions are also studied in [11,12]. More recently, the authors in [1] have proved the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type. The study of the a nonlinear boundary value problem governed by partial differential equations which describe the evolution of linear elastic materials with a nonlinear term $|u^\varepsilon|^\rho u^\varepsilon$, $\rho = p-2$ for $p>1$ has been considered in [2]. The numerical solution of the stationary Bingham fluid flow problem is studied in e.g., [8,9,13,14].

In this paper, we are interested in the asymptotic behaviour of a Bingham fluid in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$. We assume the Fourier boundary condition at the top surface and a nonlinear Tresca interface condition at the bottom one. The weak form of the problem is a variational inequality. We use the approach which consists in transposing the problem initially posed in the domain Ω^ε which depend on a small parameter ε in an equivalent problem posed in the fixed domain Ω which is independent of ε. We prove that the limit solution satisfies also a variational inequality. We further obtain a weak form of the Reynolds equation and give a lower-dimensional Bingham law, prevalent in engineering literature. In [6], the author studied a similar problem, in which, only the Direchlet conditions on the boundary have been considered.

The outline of this paper is as follows. In Section 2, basic equations and assumptions are given, also the related weak formulation and existence and uniqueness of the weak solution. In the Section 3, we establish some estimates and convergence theorem by using the Korn and Poincaré inequalities (developed recently in [4,5]). The limit problem with a specific weak form of the Reynolds equation, the uniqueness of the limit velocity distributions, and two-dimensional constitutive equation of the Bingham fluid are given in the last section.

2 Problem statement and variational formulation

Let ω be fixed region in the plane $x'=(x_1,x_2)\in\mathbb{R}^2$. We suppose that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3=\varepsilon h(x')$ where $(0<\varepsilon<1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0<\underline{h}\le h(x')\le \overline{h}$ for all $(x',0)$ in ω. We denote by Ω^ε the domain of the flow:

$$\Omega^\varepsilon = \big\{ (x',x_3)\in\mathbb{R}^3:\ (x',0)\in\omega,\ 0<x_3<\varepsilon h(x') \big\}.$$

The boundary of Ω^ε is Γ^ε. We have $\Gamma^\varepsilon=\overline{\Gamma}_1^\varepsilon\cup\overline{\Gamma}_L^\varepsilon\cup\overline{\omega}$ where Γ_L^ε is the lateral boundary.

Let σ^ε denotes the total Cauchy stress tensor:

$$\sigma^\varepsilon = -p^\varepsilon I + \sigma^{D,\varepsilon},$$