Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **6**, No. 6, pp. 797-810

DOI: 10.4208/aamm.2013.m350
December 2014

Asymptotic Analysis of a Bingham Fluid in a Thin Domain with Fourier and Tresca Boundary Conditions

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Received 17 September 2013; Accepted (in revised version) 6 June 2014

Available online 7 August 2014

Abstract. In this paper we prove first the existence and uniqueness results for the weak solution, to the stationary equations for Bingham fluid in a three dimensional bounded domain with Fourier and Tresca boundary condition; then we study the asymptotic analysis when one dimension of the fluid domain tend to zero. The strong convergence of the velocity is proved, a specific Reynolds limit equation and the limit of Tresca free boundary conditions are obtained.

AMS subject classifications: 35R35, 76F10, 78M35

Key words: Free boundary problems, Bingham fluid, asymptotic approach, Tresca law, Reynolds equation.

1 Introduction

A Bingham fluid is a rigid viscoplastic fluid that is a particular kind of non-Newtonian fluid. The name is associated to Eugene C. Bingham (1878-1945) who, for the first time, in 1916, proposed a mathematical description for this visco-plastic behaviour [3]. There are many materials in nature and industry exhibiting the behavior of the Bingham medium. For example, heavy crude oils, colloid solutions, powder mixtures, metals under pressure treatment, blood in a capillary, foodstuffs and toothpaste. The mathematical models for such materials involve the constituent law for viscous incompressible fluids with an extra stress tensor component modeling the visco-plastic effects.

The analysis of the Bingham fluid flow variational inequality was carried out in [10], where the authors investigated existence, uniqueness and regularity of the solution for

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the steady and instationary flows in a reservoir. Existence and extra regularity results for the d-dimensional Bingham fluid flow problem with Dirichlet boundary conditions are also studied in [11,12]. More recently, the authors in [1] have proved the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type. The study of the a nonlinear boundary value problem governed by partial differential equations which describe the evolution of linear elastic materials with a nonlinear term $|u^\varepsilon|^\rho u^\varepsilon$, $\rho = p-2$ for $p>1$ has been considered in [2]. The numerical solution of the stationary Bingham fluid flow problem is studied in e.g., [8,9,13,14].

In this paper, we are interested in the asymptotic behaviour of a Bingham fluid in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$. We assume the Fourier boundary condition at the top surface and a nonlinear Tresca interface condition at the bottom one. The weak form of the problem is a variational inequality. We use the approach which consists in transposing the problem initially posed in the domain Ω^ε which depend on a small parameter ε in an equivalent problem posed in the fixed domain Ω which is independent of ε. We prove that the limit solution satisfies also a variational inequality. We further obtain a weak form of the Reynolds equation and give a lower-dimensional Bingham law, prevalent in engineering literature. In [6], the author studied a similar problem, in which, only the Direchlet conditions on the boundary have been considered.

The outline of this paper is as follows. In Section 2, basic equations and assumptions are given, also the related weak formulation and existence and uniqueness of the weak solution. In the Section 3, we establish some estimates and convergence theorem by using the Korn and Poincaré inequalities (developed recently in [4,5]). The limit problem with a specific weak form of the Reynolds equation, the uniqueness of the limit velocity distributions, and two-dimensional constitutive equation of the Bingham fluid are given in the last section.

2 Problem statement and variational formulation

Let ω be fixed region in the plane $x' = (x_1, x_2) \in \mathbb{R}^2$. We suppose that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x')$ where $(0<\varepsilon<1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < \underline{h} \le h(x') \le \overline{h}$ for all $(x',0)$ in ω. We denote by Ω^ε the domain of the flow:

$$\Omega^\varepsilon = \left\{ (x', x_3) \in \mathbb{R}^3 : \ (x',0) \in \omega, \ 0 < x_3 < \varepsilon h(x') \right\}.$$

The boundary of Ω^ε is Γ^ε. We have $\Gamma^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \overline{\omega}$ where Γ_L^ε is the lateral boundary.

Let σ^ε denotes the total Cauchy stress tensor:

$$\sigma^\varepsilon = -p^\varepsilon I + \sigma^{D,\varepsilon},$$

where $\sigma^{D,\varepsilon}$ denotes its deviatoric part, and p^ε the pressure. The fluid is supposed to be viscoplastic, and the relation between $\sigma^{D,\varepsilon}$ and $D(u^\varepsilon)$ is given by the Bingham model:

$$\begin{cases} \sigma^{D,\varepsilon}=\alpha^\varepsilon\dfrac{D(u^\varepsilon)}{|D(u^\varepsilon)|}+2\mu D(u^\varepsilon), & \text{when } D(u^\varepsilon)\neq 0,\\ |\sigma^{D,\varepsilon}|\leq\alpha^\varepsilon, & \text{when } D(u^\varepsilon)=0,\end{cases}$$

or equivalently:

$$D(u^\varepsilon)=\begin{cases}\dfrac{1}{2\mu}\Big(1-\dfrac{\alpha^\varepsilon}{|\sigma^{D,\varepsilon}|}\Big)\sigma^{D,\varepsilon}, & \text{when } |\sigma^{D,\varepsilon}|>\alpha^\varepsilon,\\ 0, & \text{otherwise,}\end{cases}$$

here $\alpha^\varepsilon\geq 0$ is the yield stress, $\mu>0$ is the constant viscosity, u^ε is the velocity field and $D(u^\varepsilon)=(\nabla u^\varepsilon+(\nabla u^\varepsilon)^T)/2$.

For any tensor $\tau=(\tau_{ij})$, the notation $|\tau|$ represents the matrix norm:

$$|\tau|=\frac{1}{\sqrt{2}}\Big(\sum_{i,j}\tau_{ij}\Big)^{\frac{1}{2}}.$$

The flow of an incompressible Bingham fluid in stationary case is described by

- The law of conservation of momentum

$$-div(\sigma^\varepsilon)=f^\varepsilon \quad \text{in } \Omega^\varepsilon. \tag{2.1}$$

- The incompressibility equation

$$div(u^\varepsilon)=0 \quad \text{in } \Omega^\varepsilon. \tag{2.2}$$

Our boundary conditions is describe as

- At the surface Γ_1^ε we assume

$$\left.\begin{array}{l}\sigma_\tau(u^\varepsilon)+l^\varepsilon u^\varepsilon=0\\ u^\varepsilon\cdot n=0\end{array}\right\}\ \text{on } \Gamma_1^\varepsilon, \tag{2.3}$$

  where $l^\varepsilon>0$ on which we will bring precisions.

- On Γ_L^ε, the velocity is known and parallel to the ω-plane

$$u^\varepsilon=0 \quad \text{on } \Gamma_L^\varepsilon. \tag{2.4}$$

- On ω, there is a no-flux condition across ω so that
$$u^\varepsilon\cdot n=0, \tag{2.5}$$
the tangential velocity on ω is unknown and satisfies Tresca boundary conditions with friction coefficient k^ε [10]:
$$\left.\begin{array}{l}|\sigma_\tau^\varepsilon|<k^\varepsilon\Rightarrow u_\tau^\varepsilon=0,\\ |\sigma_\tau^\varepsilon|=k^\varepsilon\Rightarrow\exists\lambda\geq 0,\ u_\tau^\varepsilon=-\lambda\sigma_\tau^\varepsilon\end{array}\right\}\text{ on }\omega. \tag{2.6}$$
Here $n=(n_1,n_2,n_3)$ is the unit outward normal to Γ^ε, and
$$u_n^\varepsilon=u^\varepsilon\cdot n,\qquad u_\tau^\varepsilon=u^\varepsilon-u_n^\varepsilon\cdot n,$$
$$\sigma_n^\varepsilon=(\sigma^\varepsilon\cdot n)\cdot n,\qquad \sigma_\tau^\varepsilon=\sigma^\varepsilon\cdot n-(\sigma_n^\varepsilon)\cdot n,$$
are, respectively, the normal and the tangential velocity on ω, and the components of the normal and the tangential stress tensor on ω.

To get a weak formulation, we introduce:
$$K^\varepsilon=\{\phi\in H^1(\Omega^\varepsilon)^3:\ \phi=0\text{ on }\Gamma_L^\varepsilon,\ \phi\cdot n=0\text{ on }\omega\cup\Gamma_1^\varepsilon\}, \tag{2.7a}$$
$$K_{div}^\varepsilon=\{\phi\in K^\varepsilon:\ div(\phi)=0\},\quad L_0^2(\Omega^\varepsilon)=\Big\{q\in L^2(\Omega^\varepsilon):\ \int_{\Omega^\varepsilon}qdx=0\Big\}. \tag{2.7b}$$

A formal application of Green's formula, using (2.1)-(2.6) leads to the weak formulation: Find $u^\varepsilon\in K_{div}^\varepsilon$ and$p^\varepsilon\in L_0^2(\Omega^\varepsilon)$ such that
$$a(u^\varepsilon,\varphi-u^\varepsilon)-(p^\varepsilon,div\varphi)+l^\varepsilon\int_{\Gamma_1^\varepsilon}u^\varepsilon(\varphi-u^\varepsilon)+j(\phi)-j(u^\varepsilon)\geq(f^\varepsilon,\phi-u^\varepsilon),\quad\forall\varphi\in K^\varepsilon, \tag{2.8}$$
where
$$a(u^\varepsilon,\varphi-u^\varepsilon)=2\mu\int_{\Omega^\varepsilon}d_{ij}(u^\varepsilon)d_{ij}(\varphi-u^\varepsilon)dx, \tag{2.9a}$$
$$(p^\varepsilon,div\varphi)=\int_{\Omega^\varepsilon}p^\varepsilon div\varphi dx, \tag{2.9b}$$
$$j(\varphi)=\int_\omega k^\varepsilon|\varphi|dx'+\sqrt{2}\alpha^\varepsilon\int_{\Omega^\varepsilon}|D(\varphi)|dx. \tag{2.9c}$$

Theorem 2.1. *Assume that $f^\varepsilon\in L^2(\Omega^\varepsilon)^3$ and $k^\varepsilon\in L_+^\infty(\omega)$, then there exists a unique $u^\varepsilon\in K_{div}^\varepsilon$ and $p^\varepsilon\in L_0^2(\Omega^\varepsilon)$ (to an additive constant) solution to problem* (2.8)-(2.9c).

When the functions test belong to K_{div}^ε, we obtain the variational problem: Find $u\in K_{div}^\varepsilon$ such that
$$\breve{a}(u^\varepsilon,\varphi-u^\varepsilon)+j(\phi)-j(u^\epsilon)\geq(f^\varepsilon,\phi-u^\varepsilon),\quad\forall\varphi\in K_{div}^\varepsilon, \tag{2.10}$$
where
$$\breve{a}(u^\varepsilon,\varphi-u^\varepsilon)=a(u^\varepsilon,\varphi-u^\varepsilon)+l^\varepsilon\int_{\Gamma_1^\varepsilon}u^\varepsilon(\varphi-u^\varepsilon)d\tau.$$

- Using the Cauchy-Schwarz inequality we obtain that the bilinear form $\breve{a}(\cdot,\cdot)$ is continuous. There exists a constant $C(\Omega^\varepsilon)>0$:

$$|\breve{a}(\varphi,\psi)|\le(2\mu+l^\varepsilon C(\Omega^\varepsilon))|\varphi|_{1,\Omega^\varepsilon}|\psi|_{1,\Omega^\varepsilon},\quad \forall(\varphi,\psi)\in(K^\varepsilon_{div})^2.$$

- The bilinear form $\breve{a}(\cdot,\cdot)$ is coercive on $K^\varepsilon_{div}\times K^\varepsilon_{div}$. There exists [15] a constant $c>0$ such that

$$\breve{a}(\psi,\psi)\ge c|\psi|^2_{1,\Omega^\varepsilon},\quad \forall\psi\in K^\varepsilon_{div}.$$

- j is a convex, continuous and proper functional on K^ε_{div}, $\forall(\varphi,\psi)\in(K^\varepsilon_{div})^2$

$$|j(\phi)-j(\psi)|\le(|\omega|^{1/2}|k^\varepsilon|_{\infty,\omega}C(\Omega^\varepsilon)+\sqrt{2}\alpha^\varepsilon)|\varphi-\psi|_{1,\Omega^\varepsilon}.$$

  Then the existence and uniqueness of $u^\varepsilon\in K^\varepsilon_{div}$ satisfying the variational inequality (2.10) (cf. [13]).

- The proof of the existence of $p^\varepsilon\in L^2_0(\Omega^\varepsilon)$ such that $(u^\varepsilon,p^\varepsilon)$ satisfies (2.8) is given in [16].

Theorem 2.2. *Let u^ε be a solution of the variational inequality* (2.10). *We assume that $\varepsilon l^\varepsilon=\hat{l}$ and h satisfy the following condition*

$$\frac{C(\Gamma^\varepsilon_1)}{l^\varepsilon}\le\frac{1}{\mu},\tag{2.11}$$

where

$$C(\Gamma^\varepsilon_1)=2\Big\|\frac{\partial}{\partial x_2}h^\varepsilon\Big\|_{\mathcal{C}(\bar\omega)}\Big(1+\Big\|\frac{\partial}{\partial x_1}h^\varepsilon\Big\|^2_{\mathcal{C}(\bar\omega)}\Big).$$

Then

$$|\nabla(u^\varepsilon)|^2_{0,\Omega^\varepsilon}\le\Big(\frac{24\bar{h}^2\varepsilon^2}{\mu^2}+\frac{12\bar{h}\varepsilon}{l^\varepsilon\mu}\Big)|f^\varepsilon|^2_{0,\Omega^\varepsilon}.\tag{2.12}$$

Proof. From [4], we recall the following inequalies (Korn, Poincaré and Young respectively),

$$\int_{\Omega^\varepsilon}|\nabla(u^\varepsilon)|^2\le2\int_{\Omega^\varepsilon}|D(u^\varepsilon)|^2dx+\mathcal{C}(\Gamma^\varepsilon_1)\int_{\Gamma^\varepsilon_1}|u^\varepsilon|^2d\tau,\tag{2.13a}$$

$$\int_{\Omega^\varepsilon}|u^\varepsilon|^2\le2\bar{h}\varepsilon\int_{\Gamma^\varepsilon_1}|u^\varepsilon|^2+2\bar{h}^2\varepsilon^2\int_{\Omega^\varepsilon}\Big|\frac{\partial u^\varepsilon}{\partial x_3}\Big|^2,\tag{2.13b}$$

$$ab\le\eta^2\frac{a^2}{2}+\eta^{-2}\frac{b^2}{2},\quad\forall(a,b)\in\mathbb{R}^2,\ \forall\eta.\tag{2.13c}$$

Now choosing in (2.10), $\phi=0$, we get

$$a(u^\varepsilon,u^\varepsilon)+l^\varepsilon\int_{\Gamma^\varepsilon_1}|u^\varepsilon|^2d\tau+\int_\omega k^\varepsilon|u^\varepsilon|dx'+\sqrt{2}\alpha^\varepsilon\int_{\Omega^\varepsilon}|D(u^\varepsilon)|dx\le\int_{\Omega^\varepsilon}f^\varepsilon u^\varepsilon dx,\tag{2.14}$$

we have by the Cauchy-Schwarz inequality and (2.13b)

$$\begin{aligned}&\int_{\Omega^\varepsilon} f^\varepsilon u^\varepsilon dx \le |f^\varepsilon|_{0,\Omega^\varepsilon}|u^\varepsilon|_{0,\Omega^\varepsilon}\\ \le&\sqrt{2\overline{h}}\varepsilon|f^\varepsilon|_{0,\Omega^\varepsilon}\Big(\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx\Big)^{1/2}+(2(\overline{h})\varepsilon)^{1/2}|f^\varepsilon|_{0,\Omega^\varepsilon}\Big(\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2d\tau\Big)^{1/2}.\end{aligned} \tag{2.15}$$

Using Young's inequality (2.13c) for $\eta=\sqrt{\mu/2}$, $a=(\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx)^{1/2}$ and $b=\sqrt{2\overline{h}}\varepsilon|f^\varepsilon|_{0,\Omega^\varepsilon}$; then $\eta=1$, $a=\sqrt{l^\varepsilon}(\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2)^{1/2}$ and$b=\sqrt{2\overline{h}\varepsilon/l^\varepsilon}|f^\varepsilon|_{0,\Omega^\varepsilon}$, we deduce

$$\sqrt{2}\overline{h}\varepsilon|f^\varepsilon|_{0,\Omega^\varepsilon}\Big(\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx\Big)^{1/2}\le\frac{\mu}{4}\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx+\Big(\frac{2\overline{h}^2\varepsilon^2}{\mu}\Big)|f^\varepsilon|^2_{0,\Omega^\varepsilon}, \tag{2.16a}$$

$$\sqrt{2\overline{h}\varepsilon}|f^\varepsilon|_{0,\Omega^\varepsilon}\Big(\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2\Big)^{1/2}\le\frac{l^\varepsilon}{2}\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2d\tau+\frac{\overline{h}\varepsilon}{l^\varepsilon}|f^\varepsilon|^2_{0,\Omega^\varepsilon}. \tag{2.16b}$$

From (2.13a)-(2.16b), we obtain

$$\begin{aligned}&2\mu\int_{\Omega^\varepsilon}|D(u^\varepsilon)|^2dx+\frac{l^\varepsilon}{2}\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2d\tau+\sqrt{2}\alpha^\varepsilon\int_{\Omega^\varepsilon}|D(u^\varepsilon)|dx+\int_\omega k^\varepsilon|u|dx'\\ \le&\frac{\mu}{4}\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx+\Big(\frac{2\overline{h}^2\varepsilon^2}{\mu}+\frac{\overline{h}\varepsilon}{l^\varepsilon}\Big)|f^\varepsilon|^2_{0,\Omega^\varepsilon}.\end{aligned} \tag{2.17}$$

By this estimate we estimate $|D(u^\varepsilon)|^2_{0,\Omega^\varepsilon}$ and $|u^\varepsilon|^2_{0,\Gamma_1^\varepsilon}$ in (2.13b), we have

$$2\int_{\Omega^\varepsilon}|D(u^\varepsilon)|^2dx\le\frac{1}{4}\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx+\Big(\frac{2\overline{h}^2\varepsilon^2}{\mu^2}+\frac{\overline{h}\varepsilon}{l^\varepsilon\mu}\Big)|f^\varepsilon|^2_{0,\Omega^\varepsilon},$$

and

$$\mathcal{C}(\Gamma_1^\varepsilon)\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2d\tau\le\frac{\mu\mathcal{C}(\Gamma_1^\varepsilon)}{2l^\varepsilon}\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2dx+\frac{2\mathcal{C}(\Gamma_1^\varepsilon)}{l^\varepsilon}\Big(\frac{2\overline{h}^2\varepsilon^2}{\mu}+\frac{\overline{h}\varepsilon}{l^\varepsilon}\Big)|f^\varepsilon|^2_{0,\Omega^\varepsilon},$$

thus (2.12) follows from (2.11). □

3 Change of the domain and some estimates

In this section, we will use the technique of scaling in Ω^ε on the coordinate x_3, by introducing the change of the variables $z=x_3/\varepsilon$. We obtain a fixed domain Ω which is independent of ε:

We denote by $\Gamma=\overline{\Gamma}_1\cup\overline{\Gamma}_L\cup\overline{\omega}$ its boundary, then we define the following functions in Ω:

$$\hat{u}_i^\varepsilon(x',z)=u_i^\varepsilon(x',x_3),\quad i=1,2,\quad \hat{u}_3^\varepsilon(x',z)=\varepsilon^{-1}u_3^\varepsilon(x',x_3)\quad\text{and}\quad \hat{p}^\varepsilon(x',z)=\varepsilon^2p^\varepsilon(x',x_3).$$

Let us assume the following dependence (with respect of ε) of the data:

$$\hat{f}(x',z)=\varepsilon^2 f^\varepsilon(x',x_3), \quad \hat{\alpha}=\varepsilon\alpha^\varepsilon, \quad \hat{l}=\varepsilon l^\varepsilon, \quad \hat{k}=\varepsilon k^\varepsilon.$$

Let

$$\begin{aligned}
&K=\{\phi\in H^1(\Omega)^3:\hat{\phi}=0 \text{ on } \Gamma_L,\ \hat{\phi}\cdot n=0 \text{ on } \omega\cup\Gamma_1\},\\
&K_{div}=\{\hat{\phi}\in K: div(\hat{\phi})=0\},\\
&V_z=\Big\{v=(v_1,v_2)\in L^2(\Omega)^2:\frac{\partial v_i}{\partial z}\in L^2(\Omega);\ v=0 \text{ on } \Gamma_L\Big\},
\end{aligned}$$

the norm of V_z is

$$|v|_{V_z}=\Big(\sum_{i=1}^2\Big(|v_i|^2_{0,\Omega}+\Big|\frac{\partial v_i}{\partial z}\Big|^2_{0,\Omega}\Big)\Big)^{\frac{1}{2}}.$$

By injecting the new data and unknown factors in (2.8) and after multiplication by ε, we deduce

$$\begin{aligned}
&\sum_{i,j=1}^2\int_\Omega\Big[\varepsilon^2\mu\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\Big)-\hat{p}^\varepsilon\delta_{ij}\Big]\frac{\partial(\hat{\phi}_i-\hat{u}_i^\varepsilon)}{\partial x_j}dx'dz+\sum_{i=1}^2\int_\Omega\mu\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\Big)\frac{\partial(\hat{\phi}_i-\hat{u}_i^\varepsilon)}{\partial z}dx'dz\\
&+\int_\Omega\Big(2\mu\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial z}-\hat{p}^\varepsilon\Big)\frac{\partial(\hat{\phi}_3-\hat{u}_3^\varepsilon)}{\partial z}dx'dz+\sum_{j=1}^2\int_\Omega\mu\varepsilon^2\Big(\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial z}\Big)\frac{\partial(\hat{\phi}_3-\hat{u}_3^\varepsilon)}{\partial x_j}dx'dz\\
&+\sum_{i=1}^2\hat{l}\int_\omega\hat{u}_i^\varepsilon(x',h(x'))(\hat{\phi}_i(x',h(x'))-\hat{u}_i^\varepsilon(x',h(x')))\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'\\
&+\int_\omega\hat{l}\varepsilon^2\hat{u}_3^\varepsilon(x',h(x'))(\hat{\phi}_3(x',h(x'))-\hat{u}_3^\varepsilon(x',h(x')))\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'\\
&+\int_\omega\hat{k}(|\hat{\phi}-s|-|\hat{u}^\varepsilon-s|)dx'+\sqrt{2}\hat{\alpha}\int_{\Omega^\varepsilon}(|\widetilde{D}(\hat{\phi})|-|\widetilde{D}(\hat{u}^\varepsilon)|)dx'dz\\
\geq&\sum_{j=1}^2\int_{\Omega^\varepsilon}(\hat{f}_i,\hat{\phi}_i-\hat{u}_i^\varepsilon)dx'dz+\int_{\Omega^\varepsilon}\varepsilon(\hat{f}_3,\hat{\phi}_3-\hat{u}_3^\varepsilon)dx'dz,
\end{aligned}\tag{3.1}$$

where

$$|\widetilde{D}(v)|=\Big[\frac{1}{4}\sum_{i,j=1}^2\varepsilon^2\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big)^2+\frac{1}{2}\sum_{i=1}^2\Big(\frac{\partial v_i}{\partial z}+\varepsilon^2\frac{\partial v_3}{\partial x_i}\Big)^2+\varepsilon^2\Big(\frac{\partial v_3}{\partial z}\Big)^2\Big]^{\frac{1}{2}}.$$

Now we establish the estimates and convergences for the velocity field $\hat{u}^\varepsilon$ and the pressure $\hat{p}^\varepsilon$in the domain Ω.

Theorem 3.1. *Let the assumptions of Theorems 2.1 and 2.2 hold, then there exists a constant C* *independent of ε such that*

$$\varepsilon^2\sum_{i,j=1}^{2}\Big|\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\Big|_{0,\Omega}^2+\sum_{i=1}^{2}\Big|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}|_{0,\Omega}^2+\varepsilon^2\Big|\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\Big|_{0,\Omega}^2+\varepsilon^4\sum_{i=1}^{2}\Big|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}|_{0,\Omega}^2\le C, \tag{3.2a}$$

$$|\hat{u}_i^\varepsilon|_{0,\Omega}\le C \quad for\ \ i=1,2, \tag{3.2b}$$

$$|\varepsilon\hat{u}_3^\varepsilon|_{0,\Omega}\le C, \tag{3.2c}$$

$$\Big|\frac{\partial\hat{p}^\varepsilon}{\partial x_i}\Big|_{-1,\Omega}\le C \quad for\ \ i=1,2, \tag{3.2d}$$

$$\Big|\frac{\partial\hat{p}^\varepsilon}{\partial z}\Big|_{-1,\Omega}\le C\varepsilon. \tag{3.2e}$$

Proof. Passing to the fixed domain Ω in (2.12), using the new data and unknown factors, we deduce (3.2a).

To get (3.2b)-(3.2c), using (2.13b)-(2.17), we obtain

$$\varepsilon^{-1}\int_{\Omega^\varepsilon}\Omega^\varepsilon|u^\varepsilon|^2\le\frac{\mu\overline{h}}{l^\varepsilon}|\nabla u^\varepsilon|_{0,\Omega^\varepsilon}^2+\Big(\frac{8\overline{h}^3\varepsilon^2}{\mu l^\varepsilon}+\frac{4\overline{h}^2\varepsilon}{(l^\varepsilon)^2}\Big)|f^\varepsilon|_{0,\Omega^\varepsilon}^2+2\overline{h}^2\varepsilon|\nabla u^\varepsilon|_{0,\Omega^\varepsilon}^2,$$

as $\varepsilon l^\varepsilon=\hat{l}$ and $\varepsilon^3|f^\varepsilon|_{0,\Omega^\varepsilon}^2=|\hat{f}|_{0,\Omega}^2$, from (3.2a) we have

$$\sum_{i=1}^{2}|\hat{u}_i^\varepsilon|_{0,\Omega}^2+|\varepsilon\hat{u}_3^\varepsilon|_{0,\Omega}^2\le\Big(\frac{\mu\overline{h}}{\hat{l}}+2\overline{h}^2\Big)C+\Big(\frac{8\overline{h}^3}{\mu\hat{l}}+\frac{4\overline{h}^2}{\hat{l}^2}\Big)|\hat{f}|_{0,\Omega}^2,$$

hence (3.2b)-(3.2c).

For get the first estimate on the pressure in (3.2d), we choose in (3.1) $\hat{\phi}=(\hat{u}_1^\varepsilon\pm\xi,\hat{u}_2^\varepsilon,\hat{u}_3^\varepsilon)$, $\xi\in H_0^1(\Omega)$ we obtain

$$\Big|\int_\Omega\frac{\partial\hat{p}^\varepsilon}{\partial x_1}\xi dx'dz\Big|\le\mu C_1\Big(\sum_{j=1}^{2}|\frac{\partial\xi}{\partial x_j}|_{0,\Omega}+|\frac{\partial\xi}{\partial z}|_{0,\Omega}\Big)+\Big(C_2+2\hat{\alpha}\sqrt{|\Omega|}\Big)|\xi|_{1,\Omega}$$
$$+(2-\sqrt{2})\hat{\alpha}\sqrt{|\Omega|}C_1+C_3|\hat{f}|_{0,\Omega}|\xi|_{1,\Omega},$$

then (3.2d) follows for $i=1$.

When $i=2$, we choose $\hat{\phi}=(\hat{u}_1^\varepsilon,\hat{u}_2^\varepsilon\pm\xi,\hat{u}_3^\varepsilon)$. For get (3.2e), we take $\hat{\phi}=(\hat{u}_1^\varepsilon,\hat{u}_2^\varepsilon,\hat{u}_3^\varepsilon\pm\xi)$. ☐

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, there exist $u^\star=(u_1^\star,u_2^\star)\in V_z$ and $p^\star\in L_0^2(\Omega)$ such that :*

$$\hat{u}_i\rightharpoonup u_i^\star, \quad i=1,2,\ \ weakly\ in\ \ V_z, \tag{3.3a}$$

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad i,j=1,2, \ \textit{weakly in} \ \ L^2(\Omega), \tag{3.3b}$$

$$\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \ \ \textit{weakly in} \ \ L^2(\Omega), \tag{3.3c}$$

$$\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0, \quad i=1,2, \ \textit{weakly in} \ \ L^2(\Omega), \tag{3.3d}$$

$$\hat{p}^\varepsilon \rightharpoonup p^\star \ \ \textit{weakly in} \ \ L^2(\Omega), \ \ p^\star \textit{depend only of } x'. \tag{3.3e}$$

$$\varepsilon\hat{u}_3^\varepsilon \rightharpoonup 0 \ \ \textit{weakly in} \ \ L^2(\Omega). \tag{3.3f}$$

Proof. From (3.2a) and (3.2b), we deduce (3.3a). Also (3.3b) follow from (3.2a) and (3.3a). As $div(\hat{u}^\varepsilon)=0$, by (3.3b), we obtain (3.3c). From (3.2c) and (3.2a), (3.3d) holds. Using (3.2d) and (3.2e) we get (3.3e). Because $div(\hat{u}^\varepsilon)=0$, by (3.2c) and with a particular choice of test function, we get (3.3f). □

4 Study of the limit problem

In this section, we give both the equations satisfied by $p^\star$ and $u^\star$ in Ω and the inequalities for the trace of the velocity $u^\star(x',0)$ and the stress $(\partial u^\star/\partial z)(x',0)$ on $\partial\omega$.

Passing all nonlinear terms on the right and the linear terms on the left in the variational inequality (3.1). Then, we apply the $\liminf_{\varepsilon\to 0}$ on the left and the $\lim_{\varepsilon\to 0}$ on the right, using the convergence results of the Theorem 3.2, we deduce

$$\begin{aligned}
&\sum_{j=1}^{2}\mu\int_\Omega \frac{\partial u_i^\star}{\partial z}\frac{\partial(\hat{\varphi}_i-u_i^\star)}{\partial z}dx'dz-\int_\omega p^\star(x')\Big(\hat{\varphi}_1(x',h(x'))\frac{\partial h}{\partial x_1}+\hat{\varphi}_2(x',h(x'))\frac{\partial h}{\partial x_2}\Big)dx' \\
&-\int_\Omega p^\star(x')\Big(\frac{\partial\hat{\varphi}_1}{\partial x_1}+\frac{\partial\hat{\varphi}_2}{\partial x_2}\Big)dx'dz+\hat{l}\int_\omega u_i^\star(x',h(x'))[(\hat{\varphi}_i-u_i^\star)(x',h(x'))]dx' \\
&+\hat{\alpha}\int_\Omega\Big(\Big|\frac{\partial\hat{\varphi}}{\partial z}\Big|-\Big|\frac{\partial u^\star}{\partial z}\Big|\Big)dx'dz+\hat{k}\int_\omega(|\hat{\varphi}|-|u^\star|)dx'\geq\sum_{j=1}^{2}(\hat{f},\hat{\varphi}-u^\star), \quad \forall\hat{\varphi}\in\Pi(K).
\end{aligned}\tag{4.1}$$

Moreover if

$$\int_\Omega\Big(\hat{\varphi}_1(x',z))\frac{\partial\theta}{\partial x_1}(x')+\hat{\varphi}_2(x',z))\frac{\partial\theta}{\partial x_2}(x')\Big)dx'dz=0, \quad \forall\theta\in C_0^1(\omega), \tag{4.2}$$

then

$$\begin{aligned}
&\sum_{j=1}^{2}\mu\int_\Omega \frac{\partial u_i^\star}{\partial z}\frac{\partial(\hat{\varphi}_i-u_i^\star)}{\partial z}dx'dz+\hat{l}\sum_{i=1}^{2}\int_\omega u_i^\star(x',h(x'))[\hat{\varphi}_i(x',h(x')-u_i^\star(x',h(x')]dx' \\
&+\hat{\alpha}\int_\Omega\Big(\Big|\frac{\partial\hat{\varphi}}{\partial z}\Big|-\Big|\frac{\partial u^\star}{\partial z}\Big|\Big)dx'dz+\hat{k}\int_\omega(|\hat{\varphi}|-|u^\star|)dx'\geq\sum_{j=1}^{2}(\hat{f},\hat{\varphi}-u^\star),
\end{aligned}\tag{4.3}$$

where

$$\Pi(K)=\{\overline{\phi}=(\hat{\phi}_1,\hat{\phi}_2)\in H^1(\Omega)^2:\exists\hat{\phi}_3 \text{ such that } \phi=(\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3)\in K\}.$$

Lemma 4.1. *The variational inequality* (4.3) *is equivalent the following system*

$$\begin{aligned}&\mu\int_\Omega\Big|\frac{\partial u^\star}{\partial z}\Big|^2 dx'dz+\hat{l}\int_\omega |u^\star(x',h(x')|^2dx'+\hat{k}\int_\omega|u^\star|dx'\\&\quad+\hat{\alpha}\int_\Omega\Big|\frac{\partial u^\star}{\partial z}\Big|dx'dz-\int_\Omega\hat{f}u^\star dx'dz=0,\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}&\mu\int_\Omega\frac{\partial u^\star}{\partial z}\frac{\partial\hat{\phi}}{\partial z}dx'dz+\hat{l}\int_\omega(u^\star\hat{\phi})(x',h(x')dx'+\hat{k}\int_\omega|\hat{\phi}|dx'+\hat{\alpha}\int_\Omega\Big|\frac{\partial\hat{\phi}}{\partial z}\Big|dx'dz\\&\geq\int_\Omega\hat{f}\hat{\phi}dx'dz,\quad\forall\hat{\phi}\in\Sigma(K),\end{aligned}\tag{4.5}$$

where $\Sigma(K)=\{\bar{\phi}\in\Pi(K):\bar{\phi}$ *satisfies condition* (4.2)$\}$.

Proof. According to [4, Lemma 5.3], we can choose $\hat{\phi}=2u^\star$ and $\hat{\phi}=0$ respectively in (4.3), we obtain (4.4) and (4.5). Let's show reciprocally that (4.4) and (4.5) implies (4.3). For this, we consider the mapping

$$\begin{aligned}\Lambda:\ &\Sigma(K)\longrightarrow L^1(\omega)^2\times L^1(\Omega)^2,\\&\hat{\phi}\longmapsto\Lambda(\hat{\phi})=\Big(\hat{k}\hat{\phi},\frac{\partial\hat{\phi}}{\partial z}\Big).\end{aligned}$$

Let us define F by

$$F\Big(\hat{k}\hat{\phi},\frac{\partial\hat{\phi}}{\partial z}\Big)=\mu\int_\Omega\frac{\partial u^\star}{\partial z}\frac{\partial\hat{\phi}}{\partial z}dx'dz+\hat{l}\int_\omega(u^\star\hat{\phi})(x',h(x')dx'-\int_\Omega\hat{f}\hat{\phi}dx'dz,\quad\forall\hat{\phi}\in\Sigma(K).\tag{4.6}$$

For all $\hat{\psi}\in\Sigma(K)$ we choose $\hat{\phi}=\hat{\psi}$, then $\hat{\phi}=-\hat{\psi}$ in (4.5), we obtain

$$\Big|F\Big(\hat{k}\hat{\psi},\frac{\partial\hat{\psi}}{\partial z}\Big)\Big|\leq\int_\omega\hat{k}|\hat{\psi}|dx'+\hat{\alpha}\int_\Omega\Big|\frac{\partial\hat{\psi}}{\partial z}\Big|dx'dz.\tag{4.7}$$

By (4.7) F is a continuous linear functional on the subspace of $L^1(\omega)^2\times L^1(\Omega)^2$ which is the image of $\Sigma(K)$ by Λ. Then, by the Hanh-Banach theorem, there exists $(\chi,\pi)\in L^\infty(\omega)^2\times L^\infty(\Omega)^2$, with $|\chi|_{\omega,\infty}\leq 1$, $|\pi|_{\Omega,\infty}\leq 1$, such that

$$F\Big(\hat{k}\hat{\psi},\frac{\partial\hat{\psi}}{\partial z}\Big)=-\int_\omega\chi\hat{k}\hat{\psi}dx'-\hat{\alpha}\int_\Omega\pi\frac{\partial\hat{\psi}}{\partial z}dx'dz.\tag{4.8}$$

In particular, from (4.4) and (4.8), we get

$$\int_\omega\hat{k}|u^\star|dx'+\hat{\alpha}\int_\Omega\Big|\frac{\partial u^\star}{\partial z}\Big|dx'dz=\int_\omega\chi\hat{k}u^\star dx'+\hat{\alpha}\int_\Omega\pi\frac{\partial u^\star}{\partial z}dx'dz.\tag{4.9}$$

Also, from (4.6) and (4.8), we have

$$\begin{aligned}&\mu\int_\Omega \frac{\partial u^\star}{\partial z}\frac{\partial \hat{\psi}}{\partial z}dx'dz+\hat{l}\int_\omega (u^\star\hat{\psi})(x',h(x')dx'+\int_\omega \chi\hat{k}\hat{\psi}dx'\\&\quad+\hat{\alpha}\int_\Omega \pi\frac{\partial\hat{\psi}}{\partial z}dx'dz-\int_\Omega \hat{f}\hat{\psi}dx'dz=0, \quad \forall\hat{\psi}\in\Sigma(K).\end{aligned} \tag{4.10}$$

Then

$$\begin{aligned}&\mu\int_\Omega \frac{\partial u^\star}{\partial z}\frac{\partial(\hat{\psi}-u^\star)}{\partial z}dx'dz+\hat{l}\int_\omega u^\star(x',h(x'))[\hat{\psi}(x',h(x'))-u^\star(x',h(x'))]dx'\\&\quad+\hat{\alpha}\int_\Omega\Big(\Big|\frac{\partial\hat{\psi}}{\partial z}\Big|-\Big|\frac{\partial u^\star}{\partial z}\Big|\Big)dx'dz+\hat{k}\int_\omega(|\hat{\psi}|-|u^\star|)dx'-\int_\Omega \hat{f}(\hat{\psi}-u^\star)dx'dz\\&=\int_\omega \hat{k}|\hat{\psi}|dx'-\int_\omega\chi\hat{k}\hat{\psi}dx'+\hat{\alpha}\int_\Omega\Big|\frac{\partial\hat{\psi}}{\partial z}\Big|dx'dz-\hat{\alpha}\int_\Omega\pi\frac{\partial\hat{\psi}}{\partial z}dx'dz\geq 0,\end{aligned}$$

whence (4.3) follows. □

Lemma 4.2. *The solution $u^\star$ of the variational inequality* (4.3) *is unique in V_z.*

Proof. Let $u^{\star,1}$, $u^{\star,2}$ be two solutions of (4.3). Taking $\hat{\varphi}=u^{\star,2}$ and $\hat{\varphi}=u^{\star,1}$ respectively, as test functions in (4.3), we get

$$-\mu\int_\Omega\Big|\frac{\partial}{\partial z}(u^{\star,1}-u^{\star,2})\Big|^2dx'dz-\hat{l}\int_\omega|(u^{\star,1}-u^{\star,2})(x',h(x')|^2dx'\geq 0,$$

then

$$\mu\Big|\frac{\partial}{\partial z}(u^{\star,1}-u^{\star,2})\Big|^2_{0,\Omega}=0 \quad\text{and}\quad \hat{l}|u^{\star,1}-u^{\star,2}|^2_{0,\omega}=0.$$

Using Poincaré inequality, we deduce the uniqueness in V_z. □

Theorem 4.1. *Let us set*

$$\sigma^\star=-\nabla p^\star+\widetilde{\sigma}^\star \quad \textit{and} \quad \widetilde{\sigma}^\star=\mu\frac{\partial u^\star}{\partial z}+\hat{\alpha}\pi, \tag{4.11}$$

then

$$-\frac{\partial}{\partial z}\Big[\mu\frac{\partial u^\star}{\partial z}+\hat{\alpha}\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}\Big]=\hat{f}-\nabla p^\star in\ \ L^2(\Omega)^2, \tag{4.12}$$

where π finds in (4.8).

Proof. If $\partial u^\star/\partial z=0$, from (4.11) we get $|\widetilde{\sigma}^\star|\leq\hat{\alpha}$. Using now (4.9), we have

$$\hat{\alpha}\int_{|\partial u^\star/\partial z|\neq 0}\Big(\Big|\frac{\partial u^\star}{\partial z}\Big|-\pi\frac{\partial u^\star}{\partial z}\Big)dx'dz+\int_\omega\hat{k}(|u^\star|-\chi u^\star)dx'=0,$$

since $|\chi|_{\omega,\infty}\leq 1$ and $|\pi|_{\Omega,\infty}\leq 1$, we deduce

$$\Big|\frac{\partial u^\star}{\partial z}\Big|=\pi\frac{\partial u^\star}{\partial z}\quad\text{and}\quad |u^\star|=\chi u^\star. \tag{4.13}$$

Hence, if $|\partial u^\star/\partial z|\neq 0$ by (4.11) we obtain

$$\widetilde{\sigma}^\star=\mu\frac{\partial u^\star}{\partial z}+\hat{\alpha}\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}. \tag{4.14}$$

In this case

$$|\widetilde{\sigma}^\star|=\Big(\mu+\frac{\hat{\alpha}}{|\partial u^\star/\partial z|}\Big)\Big|\frac{\partial u^\star}{\partial z}\Big|=\mu|\partial u^\star/\partial z|+\hat{\alpha}>\hat{\alpha},$$

therefore, we can write

$$\mu\frac{\partial u^\star}{\partial z}=\begin{cases} 0, & \text{if } |\widetilde{\sigma}^\star|\leq\hat{\alpha},\\ \widetilde{\sigma}^\star-\hat{\alpha}\dfrac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}, & \text{if } |\widetilde{\sigma}^\star|>\hat{\alpha},\end{cases}$$

which is a lower-dimensional Bingham law.

Besides, from (4.10) there exists $p^\star\in L^2(\Omega)^2$ (see [6,14]) such that

$$\begin{aligned}&\mu\int_\Omega\frac{\partial u^\star}{\partial z}\frac{\partial\hat{\psi}}{\partial z}dx'dz+\hat{l}\int_\omega(u^\star\hat{\psi})(x',h(x')dx'+\int_\omega\underline{n}\hat{k}\hat{\psi}dx'+\hat{\alpha}\int_\Omega\underline{m}\frac{\partial\hat{\psi}}{\partial z}dx'dz-\int_\Omega\hat{f}\hat{\psi}dx'dz\\ =&-\int_\Omega\nabla p^\star\hat{\psi}dx'dz,\quad\forall\hat{\psi}\in\Pi(K).\end{aligned} \tag{4.15}$$

Using (4.14), (4.15) becomes

$$\begin{aligned}&\mu\int_\Omega\widetilde{\sigma}^\star\frac{\partial\hat{\psi}}{\partial z}dx'dz+\hat{l}\int_\omega(u^\star\hat{\psi})(x',h(x')dx'+\int_\omega\underline{n}\hat{k}\hat{\psi}dx'\\ =&\int_\Omega\hat{f}\hat{\psi}dx'dz-\int_\Omega\nabla p^\star\hat{\psi}dx'dz,\quad\forall\hat{\psi}\in\Pi(K),\end{aligned} \tag{4.16}$$

from which (4.12) follows if we take in (4.16) $\hat{\psi}\in H_0^1(\Omega)^2$. □

Theorem 4.2. *Under the assumptions of preceding theorems, the traces $s^\star$, $\tau^\star$ satisfy the following inequality*

$$\begin{aligned}&\int_\omega\Big(\frac{h^3}{12}\nabla p^\star+\widetilde{F}-\frac{h}{2}s^\star+\mu\int_0^h u^\star(x',y)dy+\hat{\alpha}\int_0^h\int_0^y\frac{\partial u^\star/\partial\xi}{|\partial u^\star/\partial\xi|}(x',\xi)d\xi dy\\ &-\frac{h}{2}\mu u^\star(x',h)-\frac{\hat{\alpha}h}{2}\int_0^h\frac{\partial u^\star/\partial\xi}{|\partial u^\star/\partial\xi|}(x',\xi)d\xi\Big)\cdot\nabla\phi(x')dx'=0,\quad\forall\widehat{\varphi}\in H^1(\omega),\end{aligned} \tag{4.17}$$

where

$$\widetilde{F}(x')=\int_0^h F(x',y)dy-\frac{h}{2}F(x',h),\quad F(x',y)=\int_0^y\int_0^\xi\widehat{f}^\varepsilon(x',t)dtd\xi.$$

Proof. We integrate twice (4.12) between 0 and z, we obtain

$$
\begin{aligned}
&-\mu u^\star(x',z)+\mu s^\star-\hat{\alpha}\int_0^z \frac{\partial u^\star/\partial\xi}{|\partial u^\star/\partial\xi|}(x',\xi)d\xi+\mu\tau^\star z+\hat{\alpha}\frac{\tau^\star}{|\tau^\star|}z \\
=&\int_0^z\int_0^\xi \hat{f}(x',y)dyd\xi-\nabla p^\star\frac{z^2}{2},
\end{aligned} \tag{4.18}
$$

in particular for $z=h$ we obtain

$$
\begin{aligned}
&-\mu u^\star(x',h)+\mu s^\star-\hat{\alpha}\int_0^h \frac{\partial u^\star/\partial\xi}{|\partial u^\star/\partial\xi|}(x',\xi)d\xi+\mu\tau^\star h+\hat{\alpha}\frac{\tau^\star}{|\tau^\star|}h \\
=&\int_0^h\int_0^\xi \hat{f}(x',y)dyd\xi-\nabla p^\star\frac{h^2}{2},
\end{aligned} \tag{4.19}
$$

integrating (4.19) between 0 and h, we get

$$
\begin{aligned}
&-\mu\int_0^h u^\star(x',y)dy+\mu s^\star h-\hat{\alpha}\int_0^h\int_0^y \frac{\partial u^\star/\partial\xi}{|\partial u^\star/\partial\xi|}(x',\xi)d\xi dy+\mu\tau^\star\frac{h^2}{2}+\hat{\alpha}\frac{\tau^\star}{|\tau^\star|}\frac{h^2}{2} \\
=&\int_0^h\int_0^y\int_0^\xi \hat{f}(x',t)dtd\xi dy-\nabla p^\star\frac{h^3}{6}.
\end{aligned} \tag{4.20}
$$

From (4.19)-(4.20), we deduce (4.17). ☐

Acknowledgments

The author would like to thank Prof. M. Boukrouche for his valuable remarks during the preparation of the paper.

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