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A Simplified Parallel Two-Level Iterative Method for Simulation of Incompressible Navier-Stokes Equations

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Abstract. Based on two-grid discretization, a simplified parallel iterative finite element method for the simulation of incompressible Navier-Stokes equations is developed and analyzed. The method is based on a fixed point iteration for the equations on a coarse grid, where a Stokes problem is solved at each iteration. Then, on overlapped local fine grids, corrections are calculated in parallel by solving an Oseen problem in which the fixed convection is given by the coarse grid solution. Error bounds of the approximate solution are derived. Numerical results on examples of known analytical solutions, lid-driven cavity flow and backward-facing step flow are also given to demonstrate the effectiveness of the method.

AMS subject classifications: 65N55, 76D07, 76M10

Key words: Navier-Stokes equations, finite element, two-level method, parallel algorithm.

1 Introduction

With the development of technology for parallel computation, parallel computing attracts more and more attentions in computational fluid dynamics community nowadays. In such parallel computing, parallel algorithms play a key role in exploiting the full potential of the computational power of parallel computers and ensuring the accuracy of the approximate solution. Therefore, much effort is thrown to the development of efficient parallel numerical methods for the Navier-Stokes equations and related problems.

Recently, based on the two-grid discretization approach of Xu and Zhou [1, 2], and motivated by the observation that for a finite element solution to the Navier-Stokes equations, low frequency components can be approximated well by a relatively coarse grid

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and high frequency components can be computed on a fine grid, some local and parallel algorithms were proposed by He et al. [3], Ma et al. [4,5], and Shang et al. [6,7]. In these algorithms, the fully nonlinear Navier-Stokes equations are first solved on a coarse grid, and then corrections are calculated locally or in parallel by solving a linear problem on a fine grid. Numerical tests showed the efficiency of the algorithms [5–8]. Furthermore, by combing classical iterative methods for the Navier-Stokes equations with this approach to local and parallel finite element computations, some parallel iterative algorithms were developed and analyzed in [9,10]. This local and parallel finite element computation approach was also combined with the variational multiscale method [11] and the subgrid stabilization method [12].

In this paper, based on two-grid finite element discretization and using domain decomposition technique, we develop a simplified parallel iterative method for the simulation of incompressible flows governed by the Navier-Stokes equations. It uses a fixed point iteration that differs from those used in [9,10] for the nonlinear Navier-Stokes equations on a coarse grid, where Stokes problems are solved, and then solves an Oseen problem in a parallel manner on a fine grid to correct the solution, where the convection term is fixed by the coarse grid solution. Compared to the methods of [9,10] where Newton and Oseen iterations were employed, this method only solves a linear Stokes problem (hence, linear with positive definite symmetric part) at each iteration. Specifically, we first iteratively solve the Navier-Stokes problem by solving a sequence of Stokes equations on a coarse grid, and then compute fine grid corrections in a parallel manner by solving a linearized Oseen problem in overlapped subdomains. This method has low communication complexity. It only requires an existing sequential solver as subproblem solver and hence can reuse existing sequential software. Under the stability condition $\frac{4N}{\nu^2}\|f\|_{-1,\Omega}<1$ (here N is defined by (2.3), ν is the kinematic viscosity of the fluid, and f the external body force exerting on the fluid), we derive the following error estimate for our parallel method:

$$\begin{aligned}&|||\nabla(u-u_m^h)|||_{0,\Omega}+|||p-p_m^h|||_{0,\Omega}\\ \le& c\big(h^s+H^{s+1}(1+\|f\|_{0,\Omega})\big)\|f\|_{s-1,\Omega}+C\Big(\frac{3N}{\nu^2}\|f\|_{-1,\Omega}\Big)^m\|f\|_{-1,\Omega},\quad 1\le s\le k,\quad m\ge 1,\end{aligned} \tag{1.1}$$

where $|||\cdot|||_{0,\Omega}$ is piecewise norm defined by (3.3), m the number of nonlinear iterations satisfying the stopping criterion, (u,p) the exact solution to the Navier-Stokes equations, (u_m^h,p_m^h) the solution obtained from our parallel finite element method, H and h are the coarse and fine grids sizes, respectively, c and C are two generic positive constants which are independent of mesh parameter and may stand for different values at their different occurrences in our paper, k and s are two positive constants related to the regularity of the solution (u,p) to the Navier-Stokes equations and the finite element spaces used for the discretization, respectively; see Theorem 3.2.

The above estimate shows that if we choose the coarse grid size H such that $H=\mathcal{O}(h^{\frac{s}{s+1}})$, then a convergence rate of the same order as the standard Galerkin finite element method in H^1-norm for the velocity and L^2-norm for the pressure can be obtained