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Note on the Stability of a Slowly Rotating Timoshenko Beam with Damping

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Abstract. This paper continues the senior author's previous investigation of the slowly rotating Timoshenko beam in a horizontal plane whose movement is controlled by the angular acceleration of the disk of the driving motor into which the beam is rigidly clamped. It was shown before that this system preserves the total energy. We consider the problem of stability of the system after introducing a particular type of damping. We show that the energy of only part of the system vanishes. We illustrate obtained solution with the critical case of the infinite value of the damping coefficient.

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1 Introduction

The stability of rotating beams has been the subject of several investigations during the last two decades. The majority of publications concentrated on Euler beam model, e.g., [1, 7, 8]. Various stability problems were in the scope of considerations in those papers. Adding damping operator to clamped–free Euler beam with shear force control model in [1] caused L^2–stability of the system. In paper [7] by J. Valverde and D. Garcia Vallejo additional effects of Coriolis forces are observed, and their influence on stability of the beam rotating with a critical angular velocity is investigated. In [8], N. Lesaffre, J. J. Sinou and F. Thouverez presented the stability analysis of a system composed of rotating beams on a flexible, circular fixed ring, using Routh-Hurwitz criterion.

Timoshenko beam model is a generalization of Euler beam model, taking into account additional rotation of a cross–section area. Again, many authors considered different stability aspects for this generalized object. In [9], M. Sabuncu and K. Evran considered

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rotating asymmetric cross–section Timoshenko beam, for which the effects of the shear coefficient, the beam length, coupling due to the center of flexure distance from the centroid and rotation on the stability are considered. In [10], S. S. Rao and R. S. Gupta investigated the rotating twisted and tapered Timoshenko beam and studied the effects on the stability of the system of twist, offset, speed of rotation and variation of depth and breadth taper ratios.

Since 1999, W. Krabs and G. Sklyar considered different controllability and stabilizability aspects of a special, undamped model of a rotating Timoshenko beam clamped to the motor disk in [3–6]. In [3], authors solved the problem of transferring the beam from a position of rest into another given position of rest within a given time. In [5], they showed how to choose a feedback control allowing to stabilize the system in a preassigned position of a rest. In [6], W. Krabs, G. Sklyar and J. Woźniak obtained conditions of exact controllability under the assumption that the physical parameter γ appearing in the model equation is rational.

In this paper, following the works [3–6], we consider the problem of stability of a slowly rotating Timoshenko beam with damping. We present different cases of a particular damping model. After a careful analysis, we show that in all considered cases, the resulting system is unstable.

The structure of the work is as follows. After Introduction, Section 2 reviews the fundamental definitions used by the transfer function method. Next, in Section 3, we introduce rotating Timoshenko beam model and analyze its stability after adding damping. In conclusions, the summary of the obtained results is given, together with the possible directions for further research.

2 Transfer function method

In this section we introduce fundamental definitions from [1,2] and explain transfer function method. We present them for self-containment of the work.

Laplace transform is an integral operator, which transforms a function $f(t)$ with a real argument $(t\geq 0)$ to a function $F(s)$ with complex argument s.

Definition 2.1. Let $f:[0,\infty)\to X$ (X-separable Hilbert space) have property that $e^{\beta t}f(t)\in L^1([0,\infty);X)$ for some real β. We call these Laplace-transformable functions and we define their Laplace transform F(s) by

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

for $s\in\overline{\mathbb{C}_\beta^+}:=\{s\in\mathbb{C}|\mathrm{Re}(s)\geq\beta\}$. Other notation of Laplace transform is $\mathcal{L}\{f(t)\}$ or $\hat{f}(s)$. In this paper we use notation $\hat{f}(s)$.

We will consider time-invariant infinite-dimensional linear systems of the form

$$\dot{z}(t)=\mathcal{A}z(t)+\mathcal{B}u(t), \quad t\geq 0, \quad z(0)=z_0, \tag{2.1a}$$
$$y(t)=\mathcal{C}z(t), \tag{2.1b}$$

where $z(t)\in Z$ is the state, $u(t)\in U$ is the input, and $y(t)\in Y$ is the output, Z, U, Y are suitable linear vector spaces, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are linear operators.

Transfer function defined below gives us a description of the answer of the system under impulse control.

Definition 2.2 (see [2])**.** Consider the state linear system (2.1) with zero initial state. If there exists a real α such that $\hat{y}(s)=G(s)\hat{u}(s)$ for $\text{Re}(s)>\alpha$, where $\hat{u}(s)$ and $\hat{y}(s)$ are the Laplace transforms of u and y, respectively, and $G(s)$ is a $\mathcal{L}(U,Y)$-valued function of a complex variable defined for $\text{Re}(s)>\alpha$, then we call $G(s)$ the transfer function of (2.1).

Transfer function method is one of methods of analysis of controllability and stability in control theory. In the simplest case of system of ordinary differential equations we can find Laplace transform of y and u, that is $\hat{y}$ and $\hat{u}$ respectively and find transfer function as $G(s)=\hat{y}(s)/\hat{u}(s)$. In this case, it is a quotient of two polynomials.

One can easily extend this method from ordinary differential equations to partial differential equations. Remind that the result of taking Laplace transforms of a partial differential equation is an ordinary differential equation. It is important to remember that in this case we also need to take Laplace transform of boundary conditions.

Main properties of a transfer function is location of poles and existence of zeros. In ordinary differential equations, zeroes (i.e., roots of the numerator) inform us about controllability–if a transfer function has a zero, then there exists a control subspace which has no influence on the system. Poles (i.e., roots of the denominator), on the other hand, inform us about stability–if they are all located on the left-half plane, then the system is stable. We can extend the transfer function based analysis for partial differential equations in a similar way, an extensive survey of this method can be found in [1].

It is worth to mention that there are many possible different definitions of stability of systems of differential equations (and responding methods of considerations) in literature–see for example [2,11,12]. Here we use the following notion of stability.

Definition 2.3 (see [1])**.** If a system maps every input u in $L^2(0,\infty)$ to an output y in $L^2(0,\infty)$ and

$$\sup_{u\neq 0}\frac{\|y\|_2}{\|u\|_2}<\infty,$$

the system is stable.

In this work we will check stability using theorem from [1].

Theorem 2.1. *A linear system is stable if and only if its transfer function G belongs to*

$$H^\infty = \Big\{ G:\mathbb{C}_0^+ \to \mathbb{C} \Big| G \text{ analytic and } \sup_{\mathrm{Re}(s)>0} |G(s)| < \infty \Big\}$$

with norm

$$\|G\|_\infty = \sup_{\mathrm{Re}(s)>0} |G(s)|.$$

In this case, we say that G is a stable transfer function.

From this moment, instead of systems of the form (2.1), we will consider linear systems of second order,

$$\begin{aligned} &\ddot{z}(t)+B\dot{z}(t)+Az(t)=Cu(t), \quad t\geq 0, \quad z(0)=z_0, \quad \dot{z}(0)=z_1, \\ &y(t)=Dz(t). \end{aligned}$$

Theorem 2.1 is also true for such systems (see [1]).

3 Rotating Timoshenko beam with damping

We consider a model from [4] of the rotation of a Timoshenko beam in a horizontal plane whose left end is rigidly clamped into the disk of a driving motor. Let r be the radius of the disk and let $\theta=\theta(t)$ be the rotation angle as a function of the time $t\geq 0$. If $w(x,t)$ denotes the deflection of the center line of the beam at the location $x\in[0,1]$ (the length of beam is assumed to be 1) and the time $t\geq 0$, and $\xi(x,t)$–the rotation angle of the cross section area at x and t and if we assume the rotation to be slow, w and ξ are governed by two partial differential equations

$$\begin{cases} \ddot{w}(x,t)-\dfrac{K}{\rho}(w''(x,t)+\xi'(x,t))=-\ddot{\theta}(t)(r+x), \\ \ddot{\xi}(x,t)-\dfrac{EA_c}{\rho}\xi''(x,t)+\frac{K}{I}(w'(x,t)+\xi(x,t))=\ddot{\theta}(t), \end{cases} \tag{3.1}$$

for $x\in(0,1)$ and $t>0$, where $\dot{w}=w_t$, $w'=w_x$, $\dot{\xi}=\xi_t$ and $\xi'=\xi_x$, K–shear modulus, ρ–linear density, E–Young's modulus, I–moment of inertia, A_c–cross section area.

The beam at $x=0$ is clamped to motor disk and at $x=1$ energy preservation law holds, from that we have boundary conditions

$$\begin{cases} w(0,t)=\xi(0,t)=0, \\ w'(1,t)+\xi(1,t)=0, \\ \xi'(1,t)=0, \end{cases} \tag{3.2}$$

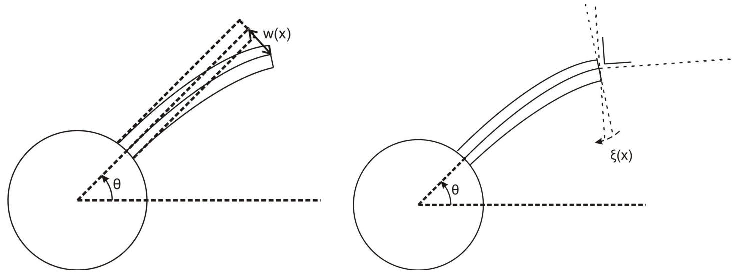

Figure 1: Deflections of the rotating beam.

for $t>0$. We denote $u(t):=\ddot{\theta}(t)$ and for simplicity we want to normalize the units, thus we will assume $EA_c/K=1$ and use an appropriate change of variables (cf. [4]) so that we may assume that

$$\frac{K}{\rho}=\frac{EA_c}{\rho}=\frac{K}{I}=1.$$

Following [3,4], we will consider operator equation of the form

$$\begin{pmatrix} \ddot{w} \\ \ddot{\xi} \end{pmatrix}+A\begin{pmatrix} w \\ \xi \end{pmatrix}=\begin{pmatrix} -r-x \\ 1 \end{pmatrix}u(t),$$

where $A:D(A)\to L^2((0,1),\mathbb{R}^2)$ is a linear operator defined by

$$A\begin{pmatrix} y \\ z \end{pmatrix}=\begin{pmatrix} -y''-z' \\ -z''+y'+z \end{pmatrix} \tag{3.3}$$

for

$$\begin{pmatrix} y \\ z \end{pmatrix}\in D(A),$$

where

$$D(A)=\left\{\begin{pmatrix} y \\ z \end{pmatrix}\in H^2((0,1),\mathbb{R}^2)\;\middle|\; \begin{matrix} y(0)=z(0)=0 \\ y'(1)+z(1)=z'(1)=0 \end{matrix}\right\}.^{\dagger}$$

Now we want to consider a rotating Timoshenko beam with damping. Again, following [4], we want to analyze a new equation of the form

$$\begin{pmatrix} \ddot{w} \\ \ddot{\xi} \end{pmatrix}+B\begin{pmatrix} \dot{w} \\ \dot{\xi} \end{pmatrix}+A\begin{pmatrix} w \\ \xi \end{pmatrix}=\begin{pmatrix} -r-x \\ 1 \end{pmatrix}u(t), \tag{3.4}$$

where $B:D(B)\to L^2((0,1),\mathbb{R}^2)$ is a symmetric linear operator with $D(A)\subseteq D(B)$. One can introduce damping operator B in a many ways. In [1] the authors considered a problem

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†Here $H^2((0,1),\mathbb{R}^2)$ denotes a Sobolev space of generalized functions $\begin{pmatrix} y \\ z \end{pmatrix}:(0,1)\to\mathbb{R}^2$, twice (weakly) differentiable, with second derivative in $L^2((0,1),\mathbb{R}^2)$.

of stability of damped Euler-Bernoulli vibrating beam, modeled by

$$\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}+C_d I\frac{\partial^5 w}{\partial x^4 \partial t}=0$$

with appropriate boundary control, where E, I are some physical constants, C_d-damping coefficient. We can rewrite it in the form similar to (3.4), i.e.,

$$\ddot{w}+B\dot{w}+Aw=0,$$

where

$$A=EI\frac{\partial^4}{\partial x^4}, \quad B=C_d I\frac{\partial^4}{\partial x^4}.$$

We will use a similar damping operator, that is

$$B\begin{pmatrix} y \\ z \end{pmatrix}=\begin{pmatrix} -\mu^2 y'' \\ 0 \end{pmatrix}, \tag{3.5}$$

where $D(B)=D(A)$. We assume the beam to be in the position of rest (with no control) at $t=0$, which leads to initial conditions of the form

$$w(x,0)=\dot{w}(x,0)=\xi(x,0)=\dot{\xi}(x,0)=0. \tag{3.6}$$

Our main problem is to analyze stability of rotating Timoshenko beam with damping. We use transfer function method and determine location of poles. To use this method we need to choose observation; we will observe a behavior of the beam at its free end (at $x=1$), we take any nontrivial linear combination of

$$y(t)=aw(1,t)+b\xi(1,t)+cw'(1,t)+d\xi'(1,t), \tag{3.7}$$

with arbitrary $a,b,c,d\in\mathbb{R}$; $a^2+b^2+c^2+d^2>0$. The main result of the paper is:

Theorem 3.1. *For any value of a damping constant $0\le\mu\le\infty$ the system*

$$\begin{cases} \ddot{w}(x,t)-\mu^2\dot{w}''(x,t)-w''(x,t)-\xi'(x,t)=-u(t)(r+x), \\ \ddot{\xi}(x,t)-\xi''(x,t)+w'(x,t)+\xi(x,t)=u(t), \end{cases}$$

is unstable.

In order to prove this we consider 3 cases: 1) $\mu=0$; 2) $\mu>0$; 3) $\mu=\infty$.

3.1 $\mu=0$

In this section we consider the trivial case with $\mu=0$, that is we assume that no damping occurs, and show how to use transfer function method to analyze stability. In the process we use different considerations than authors of [3], but we get similar results. They considered a problem of finding eigenvalues of operator A from (3.3), obtaining approximated values from some trigonometric equation. We use the transfer function method to find equivalent trigonometric equation and approximate its roots.

Let $\mu=0$, then Eqs. (3.4)-(3.5) become

$$\begin{cases} \ddot{w}(x,t)-w''(x,t)-\xi'(x,t)=-u(t)(r+x), \\ \ddot{\xi}(x,t)-\xi''(x,t)+w'(x,t)+\xi(x,t)=u(t), \end{cases} \tag{3.8}$$

for $x\in(0,1)$ and $t>0$ with boundary conditions (3.2) and initial conditions (3.6).

Theorem 3.2. *System* (3.8) *is unstable.*

First we prove:

Lemma 3.1. *Transfer function $G_1(s)$ is of the form $G_1^n(s)/G_1^d(s)$, where $G_1^n(s)$ is a numerator of this transfer function, $G_1^d(s)$ is a denominator, and*

$$\begin{aligned} G_1^d(s)=&\frac{1}{2}\Big(\cosh(\sqrt{s}\sqrt{s-i})\cosh(\sqrt{s}\sqrt{s+i}) \\ &+\frac{s}{\sqrt{s-i}\sqrt{s+i}}\sinh(\sqrt{s}\sqrt{s-i})\sinh(\sqrt{s}\sqrt{s+i})+1\Big). \end{aligned}$$

Proof. We take Laplace transform of (3.8) under assumptions (3.2) and (3.6), and obtain a system of ordinary differential equations with parameter s, of the form

$$\begin{cases} \hat{w}''(x,s)=s^2\hat{w}(x,s)-\hat{\xi}'(x,s)+\hat{u}(s)(r+x), \\ \hat{\xi}''(x,s)=(s^2+1)\hat{\xi}(x,s)+\hat{w}'(x,s)-\hat{u}(s), \end{cases} \tag{3.9}$$

with conditions

$$\begin{cases} \hat{w}(0,s)=\hat{\xi}(0,s)=0, \\ \hat{w}'(1,s)+\hat{\xi}(1,s)=0, \\ \hat{\xi}'(1,s)=0. \end{cases} \tag{3.10}$$

In order to solve system (3.9)-(3.10) we introduce a standard change of variables,

$$\begin{cases} z_1=\hat{w}, \\ z_2=\hat{\xi}, \\ z_3=\hat{w}', \\ z_4=\hat{\xi}', \end{cases} \tag{3.11}$$

and put system (3.9) into first-order form,

$$\frac{d}{dx}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s^2 & 0 & 0 & -1 \\ 0 & s^2+1 & 1 & 0 \end{bmatrix}}_{A_1(s)} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r+x \\ -1 \end{bmatrix} \hat{u}(s). \tag{3.12}$$

The matrix exponential of $A_1(s)$ with respect to x is given by

$$e^{A_1(s)x} = \begin{bmatrix} a_{11}(x,s) & a_{12}(x,s) & a_{13}(x,s) & a_{14}(x,s) \\ a_{21}(x,s) & a_{22}(x,s) & a_{23}(x,s) & a_{24}(x,s) \\ a_{31}(x,s) & a_{32}(x,s) & a_{33}(x,s) & a_{34}(x,s) \\ a_{41}(x,s) & a_{42}(x,s) & a_{43}(x,s) & a_{44}(x,s) \end{bmatrix},$$

where

$$a_{11}(x,s) = \frac{1}{2}(\cosh(\sqrt{s}\sqrt{s-i}x) + \cosh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13a}$$

$$a_{12}(x,s) = \frac{i}{2s}\Big(-\frac{(s+i)\sqrt{s-i}}{\sqrt{s}}\sinh(\sqrt{s}\sqrt{s-i}x) + \frac{(s-i)\sqrt{s+i}}{\sqrt{s}}\sinh(\sqrt{s}\sqrt{s+i}x)\Big), \tag{3.13b}$$

$$a_{13}(x,s) = \frac{1}{2s}\Big(\frac{\sqrt{s-i}}{\sqrt{s}}\sinh(\sqrt{s}\sqrt{s-i}x) + \frac{\sqrt{s+i}}{\sqrt{s}}\sinh(\sqrt{s}\sqrt{s+i}x)\Big), \tag{3.13c}$$

$$a_{14}(x,s) = \frac{i}{2s}(-\cosh(\sqrt{s}\sqrt{s-i}x) + \cosh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13d}$$

$$a_{21}(x,s) = \frac{si}{2\sqrt{s}}\Big(\frac{1}{\sqrt{s-i}}\sinh(\sqrt{s}\sqrt{s-i}x) - \frac{1}{\sqrt{s+i}}\sinh(\sqrt{s}\sqrt{s+i}x)\Big), \tag{3.13e}$$

$$a_{22}(x,s) = \frac{1}{2s}((s+i)\cosh(\sqrt{s}\sqrt{s-i}x) + (s-i)\cosh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13f}$$

$$a_{23}(x,s) = \frac{i}{2s}(\cosh(\sqrt{s}\sqrt{s-i}x) - \cosh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13g}$$

$$a_{24}(x,s) = \frac{1}{2\sqrt{s}}\Big(\frac{1}{\sqrt{s-i}}\sinh(\sqrt{s}\sqrt{s-i}x) + \frac{1}{\sqrt{s+i}}\sinh(\sqrt{s}\sqrt{s+i}x)\Big), \tag{3.13h}$$

$$a_{31}(x,s) = \frac{s}{2}\Big(\frac{\sqrt{s-i}}{\sqrt{s}}\sinh(\sqrt{s}\sqrt{s-i}x) + \frac{\sqrt{s+i}}{\sqrt{s}}\sinh(\sqrt{s}\sqrt{s+i}x)\Big), \tag{3.13i}$$

$$a_{32}(x,s) = \frac{i(s+i)(s-i)}{2s}(-\cosh(\sqrt{s}\sqrt{s-i}x) + \cosh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13j}$$

$$a_{33}(x,s) = \frac{1}{2}\Big(\frac{s-i}{s}\cosh(\sqrt{s}\sqrt{s-i}x) + \frac{s+i}{s}\cosh(\sqrt{s}\sqrt{s+i}x)\Big), \tag{3.13k}$$

$$a_{34}(x,s) = \frac{i}{2\sqrt{s}}(-\sqrt{s-i}\sinh(\sqrt{s}\sqrt{s-i}x) + \sqrt{s+i}\sinh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13l}$$

$$a_{41}(x,s)=\frac{si}{4}(\cosh(\sqrt{s}\sqrt{s-i}x)-\cosh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13m}$$

$$a_{42}(x,s)=\frac{1}{2\sqrt{s}}((s+i)\sqrt{s-i}\sinh(\sqrt{s}\sqrt{s-i}x)+(s-i)\sqrt{s+i}\sinh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13n}$$

$$a_{43}(x,s)=\frac{i}{2\sqrt{s}}(\sqrt{s-i}\sinh(\sqrt{s}\sqrt{s-i}x)-\sqrt{s+i}\sinh(\sqrt{s}\sqrt{s+i}x)), \tag{3.13o}$$

$$a_{44}(x,s)=\frac{1}{2}(\cosh(\sqrt{s}\sqrt{s-i}x)+\cosh(\sqrt{s}\sqrt{s+i}x)). \tag{3.13p}$$

General solution of system (3.12), for initial conditions $z_1(0,s)=0$, $z_2(0,s)=0$, $z_3(0,s)=\gamma(s)$ and $z_4(0,s)=\delta(s)$, is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}=e^{A_1(s)x}\begin{bmatrix} 0 \\ 0 \\ \gamma(s) \\ \delta(s) \end{bmatrix}+\int_0^x e^{A_1(s)\zeta}\begin{bmatrix} 0 \\ 0 \\ r+x-\zeta \\ -1 \end{bmatrix}\hat{u}(s)d\zeta, \tag{3.14}$$

where $\gamma(s)$, $\delta(s)$ are unknown functions. Let $x=1$, then we have

$$\begin{bmatrix} z_1(1,s) \\ z_2(1,s) \\ z_3(1,s) \\ z_4(1,s) \end{bmatrix}=e^{A_1(s)}\begin{bmatrix} 0 \\ 0 \\ \gamma(s) \\ \delta(s) \end{bmatrix}+\int_0^1 e^{A_1(s)\zeta}\begin{bmatrix} 0 \\ 0 \\ r+1-\zeta \\ -1 \end{bmatrix}\hat{u}(s)d\zeta. \tag{3.15}$$

Furthermore, we remember about

$$z_2(1,s)+z_3(1,s)=0 \quad \text{and} \quad z_4(1,s)=0. \tag{3.16}$$

Thus, we obtain

$$\begin{cases} (a_{23}(1,s)+a_{33}(1,s))\gamma(s)+(a_{24}(1,s)+a_{34}(1,s))\delta(s) \\ +\hat{u}(s)\underbrace{\int_0^1 (a_{23}(\zeta,s)+a_{33}(\zeta,s))(r+1-\zeta)-(a_{24}(\zeta,s)+a_{34}(\zeta,s))d\zeta}_{I_1(s)}=0, \\ a_{43}(1,s)\gamma(s)+a_{44}(1,s)\delta(s)+\hat{u}(s)\underbrace{\int_0^1 a_{43}(\zeta,s)(r+1-\zeta)-a_{44}(\zeta,s)d\zeta}_{I_2(s)}=0. \end{cases}$$

And after small conversion

$$\underbrace{\begin{bmatrix} a_{23}(1,s)+a_{33}(1,s) & a_{24}(1,s)+a_{34}(1,s) \\ a_{43}(1,s) & a_{44}(1,s) \end{bmatrix}}_{B_1(s)}\begin{bmatrix} \gamma(s) \\ \delta(s) \end{bmatrix}=\begin{bmatrix} -\hat{u}(s)I_1(s) \\ -\hat{u}(s)I_2(s) \end{bmatrix},$$

we can determine

$$\gamma(s)=\frac{-I_1(s)a_{44}(1,s)+I_2(s)(a_{24}(1,s)+a_{34}(1,s))}{\det B_1(s)}\hat{u}(s),$$
$$\delta(s)=\frac{-I_2(s)(a_{23}(1,s)+a_{33}(1,s))+I_1(s)a_{43}(1,s)}{\det B_1(s)}\hat{u}(s).$$

Remind that we observe the beam at its free end, that is as observation we take nontrivial linear combination (3.7), thus output $\hat{y}(s)$ reads $\hat{y}(s)=a\hat{w}(1,s)+b\hat{\xi}(1,s)+c\hat{w}'(1,s)+d\hat{\xi}'(1,s)=az_1(1,s)+bz_2(1,s)+cz_3(1,s)+dz_4(1,s)$. Hence, we can obtain formula for $z_1(1,s)$ as

$$z_1(1,s)=a_{13}(1,s)\gamma(s)+a_{14}(1,s)\delta(s)+\hat{u}(s)\int_0^1 a_{13}(\zeta,s)(r+1-\zeta)-a_{14}(\zeta,s)d\zeta,$$

and for $z_2(1,s)$, $z_3(1,s)$, $z_4(1,s)$ analogously. It is easy to see that for any such choice of observation, transfer function $G_1(s)$ is a fraction, where denominator is equal to $\det(B_1(s))$. Moreover, using (3.13) one can calculate

$$\begin{aligned}\det(B_1)=&\frac{1}{2}\Big(\cosh(\sqrt{s}\sqrt{s-i})\cosh(\sqrt{s}\sqrt{s+i})\\&+\frac{s}{\sqrt{s-i}\sqrt{s+i}}\sinh(\sqrt{s}\sqrt{s-i})\sinh(\sqrt{s}\sqrt{s+i})+1\Big),\end{aligned}\tag{3.17}$$

which finishes the proof of Lemma 3.1. □

Now we proceed with the proof of Theorem 3.2.

Proof of Theorem 3.2. We will assume that nominator and denominator of transfer function do not cancel, which can be guaranteed provided that r is non-singular radius (see [5,6]).

Let $s=it$, then determination of nonzero poles of $G_1(s)$ is equivalent to solving

$$\cos(\sqrt{t^2-t})\cos(\sqrt{t^2+t})-\frac{t}{\sqrt{t^2-1}}\sin(\sqrt{t^2-t})\sin(\sqrt{t^2+t})+1=0.\tag{3.18}$$

Observe that all zeroes of (3.18) are real, therefore all zeros of $\det(B_1)=0$ (poles of $G_1(s)$) are on the imaginary axis Re $s=0$. For sufficiently large t we have

$$t\approx\frac{2k+1}{2}\pi,$$

and

$$s\approx i\frac{2k+1}{2}\pi.$$

Using Theorem 2.1 we can say that system (3.8) is unstable, which finishes the proof. □

The result which we received is similar to results of [3] (see Lemma 2.2 therein), although a different notion of stability is consider there.

3.2 $\mu>0$

This section is devoted to the main problem of this paper, i.e., to analyzing stability of Timoshenko beam with nonzero damping which was introduced by (3.5).

Let $\mu>0$, Eqs. (3.4)-(3.5) take form

$$\begin{cases} \ddot{w}(x,t)-\mu^2\dot{w}''(x,t)-w''(x,t)-\xi'(x,t)=-u(t)(r+x), \\ \ddot{\xi}(x,t)-\xi''(x,t)+w'(x,t)+\xi(x,t)=u(t), \end{cases} \tag{3.19}$$

for $x\in(0,1)$ and $t>0$ with boundary conditions (3.2) and initial conditions (3.6).

Theorem 3.3. *System* (3.19) *is unstable.*

To prove it we need the following.

Lemma 3.2. *Transfer function $G_2(s)$ is of the form $G_2^n(s)/G_2^d(s)$, where $G_2^n(s)$ is a numerator of this transfer function, $G_2^d(s)$ is a denominator. For sufficiently large $|s|$, $G_2^d(s)$ can be approximated by*

$$G_2^d(s)\approx\cosh\Big(\frac{\sqrt{s}}{\mu}\Big)\cosh(s)+\frac{1}{\mu s^{\frac{5}{2}}}\sinh\Big(\frac{\sqrt{s}}{\mu}\Big)\sinh(s).$$

Proof. After taking Laplace transform of (3.19) under assumptions (3.2) and (3.6), we obtain

$$\begin{cases} \hat{w}''(x,s)=\dfrac{s^2}{1+\mu^2 s}\hat{w}(x,s)-\dfrac{1}{1+\mu^2 s}\hat{\xi}'(x,s)+\dfrac{1}{1+\mu^2 s}\hat{u}(s)(r+x), \\ \hat{\xi}''(x,s)=(s^2+1)\hat{\xi}(x,s)+\hat{w}'(x,s)-\hat{u}(s), \end{cases}$$

with conditions (3.10) like previously. Again we change variables as

$$\begin{cases} v_1=\hat{w}, \\ v_2=\hat{\xi}, \\ v_3=\hat{w}', \\ v_4=\hat{\xi}', \end{cases}$$

and put system (3.19) into first-order form as before, that is

$$\frac{d}{dx}\begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}=\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \dfrac{s^2}{1+\mu^2 s} & 0 & 0 & -\dfrac{1}{1+\mu^2 s}\\ 0 & s^2+1 & 1 & 0\end{bmatrix}}_{A_2(s)}\begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}+\begin{bmatrix} 0\\ 0\\ \dfrac{r+x}{1+\mu^2 s}\\ -1\end{bmatrix}\hat{u}(s). \tag{3.20}$$

Denote matrix exponential of $A_2(s)$ as

$$e^{A_2(s)x} = \begin{bmatrix} b_{11}(x,s) & b_{12}(x,s) & b_{13}(x,s) & b_{14}(x,s) \\ b_{21}(x,s) & b_{22}(x,s) & b_{23}(x,s) & b_{24}(x,s) \\ b_{31}(x,s) & b_{32}(x,s) & b_{33}(x,s) & b_{34}(x,s) \\ b_{41}(x,s) & b_{42}(x,s) & b_{43}(x,s) & b_{44}(x,s) \end{bmatrix}.$$

Detailed formulas for b_{ij}'s are very complicated and will be omitted as long as possible. Doing steps (3.14)-(3.16) as previous, we obtain

$$\begin{cases} \big(b_{23}(1,s)+b_{33}(1,s)\big)\gamma(s)+\big(b_{24}(1,s)+b_{34}(1,s)\big)\delta(s) \\ +\hat{u}(s)\underbrace{\int_0^1 \big(b_{23}(\zeta,s)+b_{33}(\zeta,s)\big)\frac{r+1-\zeta}{1+\mu^2 s}-\big(b_{24}(\zeta,s)+b_{34}(\zeta,s)\big)d\zeta}_{I_3(s)}=0, \\ b_{43}(1,s)\gamma(s)+b_{44}(1,s)\delta(s)+\hat{u}(s)\underbrace{\int_0^1 b_{43}(\zeta,s)\frac{r+1-\zeta}{1+\mu^2 s}-b_{44}(\zeta,s)d\zeta}_{I_4(s)}=0, \end{cases}$$

and after small conversion we get

$$\underbrace{\begin{bmatrix} b_{23}(1,s)+b_{33}(1,s) & b_{24}(1,s)+b_{34}(1,s) \\ b_{43}(1,s) & b_{44}(1,s) \end{bmatrix}}_{B_2(s)}\begin{bmatrix} \gamma(s) \\ \delta(s) \end{bmatrix}=\begin{bmatrix} -\hat{u}(s)I_3(s) \\ -\hat{u}(s)I_4(s) \end{bmatrix}.$$

Again we observe the beam at its free end, that is $y(t)=aw(1,t)+b\xi(1,t)+cw'(1,t)+d\xi'(1,t)$, thus denominator of transfer function $G_2(s)$ of system (3.19) can be written as

$$\det(B_2)=\sigma_3+\sigma_4\cosh(\sigma_1)\cosh(\sigma_2)+\sigma_5\sinh(\sigma_1)\sinh(\sigma_2),$$

where

$$\begin{cases} \sigma_1(s)=\dfrac{\sqrt{s}\sqrt{\mu^2+s(2+s\mu^2)-\sqrt{-4+(1+s^2)^2\mu^4}}}{\sqrt{2+2s\mu^2}}, \\ \sigma_2(s)=\dfrac{\sqrt{s}\sqrt{\mu^2+s(2+s\mu^2)+\sqrt{-4+(1+s^2)^2\mu^4}}}{\sqrt{2+2s\mu^2}}, \\ \sigma_3(s)=\dfrac{-2+\mu^4+s^2\mu^4}{-4+(1+s^2)^2\mu^4}, \\ \sigma_4(s)=\dfrac{-2+s^2\mu^4+s^4\mu^4}{-4+(1+s^2)^2\mu^4}, \\ \sigma_5(s)=\dfrac{-4s+2\mu^2-2s^2\mu^2+2s\mu^4+2s^3\mu^4}{(-4+(1+s^2)^2\mu^4)\sqrt{4s\mu^2+4s^2+4s^3\mu^2+4}}. \end{cases}$$

Note that for $\mu=0$ our results coincide with those from previous section, namely Eq. (3.17). As we can see, analyzing $\det(B_2(s))$ is nontrivial, we will use asymptotic behavior method from [4]. Then for sufficiently large s we obtain

$$\begin{cases} \sigma_1(s)=\frac{1}{\mu}\sqrt{s}-\frac{1}{2\mu^3}\frac{1}{\sqrt{s}}+\varphi_1\Big(\frac{1}{s}\Big), \\ \sigma_2(s)=s+\frac{1}{2}\frac{1}{s}+\varphi_2\Big(\frac{1}{s^2}\Big), \\ \sigma_3(s)=\frac{1}{s^2}+\Big(-1-\frac{2}{\mu^4}\Big)\frac{1}{s^4}+\Big(1+\frac{8}{\mu^4}\Big)\frac{1}{s^6}+\varphi_3\Big(\frac{1}{s^7}\Big), \\ \sigma_4(s)=1+\Big(-2+\frac{1}{\mu^2}\Big)\frac{1}{s^2}+\Big(3+\frac{2}{\mu^4}-\frac{2}{\mu^2}\Big)\frac{1}{s^4}+\varphi_4\Big(\frac{1}{s^5}\Big), \\ \sigma_5(s)=\frac{1}{\mu}\frac{1}{s^{\frac{5}{2}}}-\frac{3}{2\mu^3}\frac{1}{s^{\frac{7}{2}}}-\frac{3(3+4\mu^4)}{8\mu^5}\frac{1}{s^{\frac{9}{2}}}+\varphi_5\Big(\frac{1}{s^5}\Big), \end{cases}$$

where each $\varphi_i(\cdot)$ is an analytic function in a neighbourhood of 0 with

$$\lim_{|s|\to\infty}\varphi_i\left(\frac{1}{s}\right)=0.$$

It follows that

$$\begin{cases} \widetilde{\sigma}_1(s)\sim\frac{1}{\mu}\sqrt{s}, \\ \widetilde{\sigma}_2(s)\sim s, \\ \widetilde{\sigma}_3(s)\sim 0 \text{ as } |s|\to\infty, \\ \widetilde{\sigma}_4(s)\sim 1, \\ \widetilde{\sigma}_5(s)\sim\frac{1}{\mu s^{\frac{5}{2}}}. \end{cases}$$

And we obtain an approximate expression

$$\cosh\Big(\frac{\sqrt{s}}{\mu}\Big)\cosh(s)+\frac{1}{\mu s^{\frac{5}{2}}}\sinh\Big(\frac{\sqrt{s}}{\mu}\Big)\sinh(s),$$

which finishes the proof of Lemma 3.2. □

Now we go back to the proof of Theorem 3.3.

Proof of Theorem 3.3. Instead of equation $\det(B_2(s))=0$ we consider an approximate equation, in the form

$$\cosh\Big(\frac{\sqrt{s}}{\mu}\Big)\cosh(s)+\frac{1}{\mu s^{\frac{5}{2}}}\sinh\Big(\frac{\sqrt{s}}{\mu}\Big)\sinh(s)\approx 0. \tag{3.21}$$

After small calculations we can rewrite it approximately as

$$(1+e^{-2\frac{\sqrt{s}}{\mu}})(1+e^{2s})\approx 0.$$

Solution of (3.21) consist of two families, $\{s_k^1\}_{k=k_0}^{\infty}$ and $\{s_k^2\}_{k=k_0}^{\infty}$, where

$$s_k^1\approx\frac{\pi i}{2}(2k+1),\quad s_k^2\approx -\mu^2\Big(\frac{\pi}{2}+k\pi\Big)^2 \text{ as } k\to\infty.$$

Using Theorem 2.1 we obtain that system (3.19) is unstable. □

Corollary 3.1. *First family of poles* $\{s_k^1\}$ *in* (3.19) *is (approximately) the same like in undamped Timoshenko beam equation, while second family of poles* $\{s_k^2\}$ *is (approximately) the same like in heat flow equation. Adding damping operator*

$$B\begin{pmatrix} y \\ z \end{pmatrix}=\begin{pmatrix} -\mu^2 y'' \\ 0 \end{pmatrix}$$

causes a phenomenon of a heat generation type.

3.3 $\mu=\infty$

Now we consider limit case $\mu=\infty$. Remind that for any $\mu>0$, we have

$$\begin{cases} \ddot{w}(x,t)-\mu^2\dot{w}''(x,t)-w''(x,t)-\xi'(x,t)=-u(t)(r+x), \\ \ddot{\xi}(x,t)-\xi''(x,t)+w'(x,t)+\xi(x,t)=u(t), \end{cases}$$

for $x\in(0,1)$ and $t>0$ with boundary conditions (3.2) and initial conditions (3.6).

Dividing first of these equations by μ^2 and passing with $\mu\to\infty$, we obtain

$$\begin{cases} \dot{w}''(x,t)=0, \\ \ddot{\xi}(x,t)-\xi''(x,t)+w'(x,t)+\xi(x,t)=u(t), \end{cases} \tag{3.22}$$

for $x\in(0,1)$ and $t>0$ with boundary conditions (3.2) and initial conditions

$$w(x,0)=\xi(x,0)=\dot{\xi}(x,0)=0. \tag{3.23}$$

Theorem 3.4. *System* (3.22) *is unstable.*

We prove the following lemma.

Lemma 3.3. *Transfer function* $G_3(s)$ *is of the form* $G_3^n(s)/G_3^d(s)$*, where* $G_3^n(s)$ *is a numerator of this transfer function,* $G_3^d(s)$ *is a denominator. For sufficiently large* $|s|$*,* $G_3^d(s)$ *can be approximated by*

$$G_3^d(s)\approx\cosh(s).$$

Proof. After taking Laplace transform of (3.22) under assumptions (3.2) and (3.23), we obtain

$$\begin{cases} s\hat{w}''(x,s)=0, \\ \hat{\xi}''(x,s)=(s^2+1)\hat{\xi}(x,s)+\hat{w}'(x,s)-\hat{u}(s), \end{cases}$$

with conditions (3.10) like previously. We change variables as before in (3.11),

$$\begin{cases} q_1=\hat{w}, \\ q_2=\hat{\xi}, \\ q_3=\hat{w}', \\ q_4=\hat{\xi}', \end{cases}$$

and put differential equations into first-order form

$$\frac{d}{dx}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}=\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & s^2+1 & 1 & 0 \end{bmatrix}}_{A_3(s)}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}\hat{u}(s). \tag{3.24}$$

The matrix exponential of $A_3(s)$ is given by

$$e^{A_3(s)x}=\begin{bmatrix} 1 & 0 & x & 0 \\ 0 & \cosh(\sqrt{s^2+1}x) & \dfrac{\cosh(\sqrt{s^2+1}x)-1}{s^2+1} & \dfrac{\sinh(\sqrt{s^2+1}x)}{\sqrt{s^2+1}} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{s^2+1}\sinh(\sqrt{s^2+1}x) & \dfrac{\sinh(\sqrt{s^2+1}x)}{\sqrt{s^2+1}} & \cosh(\sqrt{s^2+1}x) \end{bmatrix}.$$

Doing steps similar to (3.14)-(3.16), we obtain

$$\begin{cases} \dfrac{s^2+\cosh(\sqrt{s^2+1})}{s^2+1}\gamma(s)+\dfrac{\sinh(\sqrt{s^2+1})}{\sqrt{s^2+1}}\delta(s)=0, \\ \dfrac{\sinh(\sqrt{s^2+1})}{\sqrt{s^2+1}}\gamma(s)+\cosh(\sqrt{s^2+1})\delta(s)+\hat{u}(s)\underbrace{\int_0^1\cosh(\sqrt{s^2+1}\zeta)d\zeta}_{I_5(s)}=0. \end{cases}$$

After small conversion we get

$$\underbrace{\begin{bmatrix} \dfrac{s^2+\cosh(\sqrt{s^2+1})}{s^2+1} & \dfrac{\sinh(\sqrt{s^2+1})}{\sqrt{s^2+1}} \\ \dfrac{\sinh(\sqrt{s^2+1})}{\sqrt{s^2+1}} & \cosh(\sqrt{s^2+1}) \end{bmatrix}}_{B_3(s)}\begin{bmatrix} \gamma(s) \\ \delta(s) \end{bmatrix}=\begin{bmatrix} 0 \\ -\hat{u}(s)I_5(s) \end{bmatrix}.$$

Hence, denominator of transfer function $G_3(s)$ of system (3.24) is of the form

$$G_3^d(s) = \det(B_3(s)) = \frac{s^2\cosh(\sqrt{s^2+1})+1}{s^2+1}. \tag{3.25}$$

For sufficiently large $|s|$, formula (3.25) can be approximated by

$$\cosh(s),$$

which finishes the proof of Lemma 3.3. □

Now we proceed with the proof of Theorem 3.4.

Proof of Theorem 3.4. We consider an approximate equation, in the form

$$\cosh(s) \approx 0. \tag{3.26}$$

Solution of (3.26) is

$$s \approx \frac{\pi i}{2}(2k+1).$$

So, we complete the proof. □

Observe that all poles of $G_3(s)$ lie on imaginary axis. From Theorem 2.1, we can say that system (3.22) is unstable. Furthermore, comparing this with previous result we see that family $\{s_k^2\}$ escapes to $-\infty+0i$.

4 Conclusions

We analyzed stability of a slowly rotating Timoshenko beam in a horizontal plane, rigidly clamped to the motor disk, observed at the right end. After taking into account the special damping, the only part of the energy of the system vanishes. In the limit case of infinite unrealistic damping we observe the same behaviour.

In this work we assumed that the damping operator had the special form, namely

$$B\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\mu^2 y'' \\ 0 \end{pmatrix}.$$

In the sequel we consider a more general damping operator of the form

$$B\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\mu^2 y'' \\ \nu^2 z \end{pmatrix}.$$

Nomenclature

A	$=$	operator of movement
A_c	$=$	cross section area
B	$=$	damping operator
E	$=$	Young's modulus
$\hat{f}(s)$	$=$	Laplace transform
I	$=$	moment of inertia
K	$=$	shear modulus
$\xi(x,t)$	$=$	rotation angle of cross section area
$G(s)$	$=$	transfer function
ρ	$=$	linear density
$\theta(t)$	$=$	rotation angle
t	$=$	time
$u(t)$	$=$	input (control)
$w(x,t)$	$=$	deflection of the center line
x	$=$	coordinate along the beam
$y(t)$	$=$	output

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