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Implicit DG Method for Time Domain Maxwell's Equations Involving Metamaterials

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Abstract. An implicit discontinuous Galerkin method is introduced to solve the time-domain Maxwell's equations in metamaterials. The Maxwell's equations in metamaterials are represented by integral-differential equations. Our scheme is based on discontinuous Galerkin method in spatial domain and Crank-Nicolson method in temporal domain. The fully discrete numerical scheme is proved to be unconditionally stable. When polynomial of degree at most p is used for spatial approximation, our scheme is verified to converge at a rate of $\mathcal{O}(\tau^2+h^{p+1/2})$. Numerical results in both 2D and 3D are provided to validate our theoretical prediction.

AMS subject classifications: 65M12, 65M60

Key words: Maxwell's equations, metamaterials, fully disctete, DG method, L^2-stability, L^2-error estimate.

1 Introduction

The metamaterials are artificially structured electromagnetic materials. It has some exotic properties, such as negative refractive index and amplification of evanescent waves, which may not be found in nature. Since it was first constructed in 2000, there are a number of works on the study of metamaterials and their applications in different areas.

To our knowledge, the numerical simulation of metamaterials plays an important role in the design of new metamaterials and discovery of new phenomenon of them [5]. Among them, the widely used numerical methods for the simulation of metamaterials are finite difference time domain method (FDTD) [9, 24], finite element method (FEM) [19] and the commercial packages, such as HFSS and COMSOL et al..

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Since the discontinuous Galerkin (DG) method was first proposed in 1973 [23], it has become one of the most popular methods for solving various partial differential equations [2,11]. Actually, the DG method uses discontinuous piecewise polynomials as both trial and test functions. In this approach, the discontinuities at the element interfaces allow the design of suitable inter-element boundary treatments to obtain highly accurate and stable methods in many difficult situations. It is well known that the DG method has several distinctive advantages, e.g., applicability for non-conforming mesh, high-order accuracy, flexibility in handling material interface and high parallelizability. We refer to the survey papers [1,4] and the books [6,11] and their references therein for more details about it.

The DG methods have been investigated for Maxwell's equations in both free space [3, 7, 8, 27] and dispersive media [12, 14, 15, 18, 21, 25, 26] whose permittivity depends on the wave frequency. In [25], Wang et al. introduce a semi-discrete divergence-free DG method for solving Maxwell's equations in dispersive media under a unified framework. It is proved that the convergence rate of the semi-discrete method is $\mathcal{O}(h^{p+1/2})$. Actually the discretization of the spatial domain, leads to a Volterra integro-differential system in time t. Then a continuous Galerkin method is used to solve this reduced system. In [27], Xie et al. develop an unconditionally stable space-time DG method for solving Maxwell's equations in free space and obtain the convergence rate of $\mathcal{O}(\tau^{p+1}+h^{p+1/2})$ in the L^2-norm when the polynomials of degree at most p are used in both temporal and spatial discretization. In [26], Wang et al. extend this space-time DG method to dispersive media and give both the theoretical analysis and numerical examples.

Recently, Li in [16] develop a DG method in space for solving Maxwell's equations in metamaterials and perfectly matched layers with Runge-Kutta method in time. In it several numerical examples were given to show that this method is efficient, but the theoretical analysis is missing. In [20], Li et al. develop a leap-frog type DG method for solving the time-domain Maxwell's equations in metamaterials, and provide both the stability and convergence analysis. In [18], Li et al. develope a leap-frog type DG method for solving the time-domain Maxwell's equations in metamaterials based on an auxiliary differential equations (ADE) method, and prove that, under some CFL condition, this method is stable and convergent.

The main aim of this paper is to develop a new scheme for solving time-domain Maxwell's equations in metamaterials. The model problem is the first-order Maxwell's equations in metamaterials where the dispersive character is taken into account via an integral term. Our scheme is based on discontinuous Galerkin method in spatial domain and Crank-Nicolson method in temporal domain. Then the unconditional stability and convergence rate of $\mathcal{O}(\tau^2+h^{p+1/2})$ are obtained for our scheme. Theoretical results are validated by some numerical examples.

The rest of this paper is organized as follows. In Section 2, we present the governing equations for metamaterials. The fully-discrete scheme is introduced in Section 3. Both the L^2-stability and L^2-error estimate are proved in Section 4. In Section 5, some numerical results are provided to support our theory analysis.

2 The governing equation

The governing Maxwell's equations in metamaterials, which is defined in $\Omega\times[0,T]$, can be written as follows [18],

$$\epsilon_0\frac{\partial\mathbf{E}}{\partial t}=\nabla\times\mathbf{H}-\mathbf{J}, \tag{2.1a}$$

$$\mu_0\frac{\partial\mathbf{H}}{\partial t}=-\nabla\times\mathbf{E}-\mathbf{K}, \tag{2.1b}$$

$$\frac{\partial\mathbf{J}}{\partial t}+\Gamma_e\mathbf{J}=\epsilon_0\omega_{pe}^2\mathbf{E}, \tag{2.1c}$$

$$\frac{\partial\mathbf{K}}{\partial t}+\Gamma_m\mathbf{K}=\mu_0\omega_{pm}^2\mathbf{H}, \tag{2.1d}$$

where ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively, ω_{pe} and ω_{pm} are the electric and magnetic plasma frequency, respectively, Γ_e and Γ_m are the electric and magnetic damping frequency, respectively. For simplicity, we assume that Ω is a bounded and convex domain with the boundary of $\partial\Omega$ perfect conducting, i.e.,

$$\mathbf{n}\times\mathbf{E}=0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. The initial conditions for the system (2.1a) to (2.1d) are assumed to be

$$\mathbf{H}(\mathbf{x},0)=\mathbf{H}_0(\mathbf{x}),\quad \mathbf{E}(\mathbf{x},0)=\mathbf{E}_0(\mathbf{x}),\quad \mathbf{K}(\mathbf{x},0)=\mathbf{K}_0(\mathbf{x}),\quad \mathbf{J}(\mathbf{x},0)=\mathbf{J}_0(\mathbf{x}). \tag{2.3}$$

By (2.1c) and (2.1d), we get

$$\mathbf{J}(\mathbf{x},t;\mathbf{E})=e^{-\Gamma_e t}\mathbf{J}_0(\mathbf{x})+\epsilon_0\omega_{pe}^2\int_0^t e^{-\Gamma_e(t-s)}\mathbf{E}(\mathbf{x},s)ds\equiv e^{-\Gamma_e t}\mathbf{J}_0(\mathbf{x})+\mathcal{J}\mathbf{E}, \tag{2.4a}$$

$$\mathbf{K}(\mathbf{x},t;\mathbf{H})=e^{-\Gamma_m t}\mathbf{K}_0(\mathbf{x})+\mu_0\omega_{pm}^2\int_0^t e^{-\Gamma_m(t-s)}\mathbf{H}(\mathbf{x},s)ds\equiv e^{-\Gamma_e t}\mathbf{K}_0(\mathbf{x})+\mathcal{K}\mathbf{H}. \tag{2.4b}$$

Thus, (2.1a) and (2.1b) can be written as

$$\epsilon_0\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{H}+\epsilon_0\omega_{pe}^2\int_0^t e^{-\Gamma_e(t-s)}\mathbf{E}(x,s)ds=-e^{-\Gamma_e t}\mathbf{J}_0(x), \tag{2.5a}$$

$$\mu_0\frac{\partial\mathbf{H}}{\partial t}+\nabla\times\mathbf{E}+\mu_0\omega_{pm}^2\int_0^t e^{-\Gamma_m(t-s)}\mathbf{H}(x,s)ds=-e^{-\Gamma_m t}\mathbf{K}_0(x). \tag{2.5b}$$

In the following, $(H^k(\Omega))^3$ denotes the standard Sobolev space equipped with the norm $\|\cdot\|_k$ and seminorm $|\cdot|_k$. In particular, $\|\cdot\|_0$ is the $(L^2(\Omega))^3$-norm. Moreover, for

$$\mathbf{Q}=\begin{pmatrix}\epsilon_0 & 0\\ 0 & \mu_0\end{pmatrix},\quad \mathbf{u}=(\mathbf{E},\mathbf{H})^T,$$

with $\mathbf{E}(\cdot,t),\mathbf{H}(\cdot,t)\in(H^k(\Omega))^3$, for $\forall t\in(0,T)$, we define

$$\|\mathbf{u}(\cdot,t)\|_{k,\Omega}=\left(\|\mathbf{E}(\cdot,t)\|_{k,\Omega}^2+\|\mathbf{H}(\cdot,t)\|_{k,\Omega}^2\right)^{1/2},$$

and

$$\|\mathbf{Q}^{1/2}\mathbf{u}(\cdot,t)\|_{k,\Omega}=\left(\epsilon_0\|\mathbf{E}(\cdot,t)\|_{k,\Omega}^2+\mu_0\|\mathbf{H}(\cdot,t)\|_{k,\Omega}^2\right)^{1/2}.$$

3 Numerical scheme

We assume that Ω is decomposed by a regular tetrahedra mesh $\mathcal{T}_h$ with maximum diameter h. An element is denoted by K, a face by e, and the outward unit normal by $\mathbf{n}_K$. We also denote by $\mathcal{E}_\mathcal{I}$ the union of all interior faces of $\mathcal{T}_h$, by $\mathcal{E}_\mathcal{D}$ the union of all boundary faces of $\mathcal{T}_h$, and by $\mathcal{E}=\mathcal{E}_\mathcal{I}\bigcup\mathcal{E}_\mathcal{D}$ the union of all faces of $\mathcal{T}_h$. Moreover, let $0=t_0<t_1<\cdots<t_N=T$ be a subdivition of the interval $I=[0,T]$ with elements denoted by $I_j=[t_j,t_{j+1}]$, $j=0,1,\cdots,N-1$, where $\tau=T/N$ and $t_j=j\tau$.

Let $P^p(K)$ denote the space of polynomials with degree at most p. Then the DG finite element space for the spatial discretization is given by

$$\mathbf{V}_h^p=\bar{V}_h^p\oplus\bar{V}_h^p,$$

where

$$\bar{V}_h^p=\{v\in(L^2(\Omega))^3:v|_K\in(P^p(K))^3,\ K\in\mathcal{T}_h\}.$$

It is known that the choice of the numerical flux plays a significant role in the definition of DG scheme. For this purpose, we need to introduce some notations first. Let e be an interior face belonging to element K. Set

$$\mathbf{v}^{int(K)}(\mathbf{x})=\lim_{\delta\to0-}\mathbf{v}(\mathbf{x}+\delta\mathbf{n}_K),\quad \mathbf{v}^{ext(K)}(\mathbf{x})=\lim_{\delta\to0+}\mathbf{v}(\mathbf{x}+\delta\mathbf{n}_K),\quad \forall\mathbf{x}\in e.$$

The average and tangential jump of $\mathbf{v}$ on any interior face $e\subset\mathcal{E}_\mathcal{I}$ are defined as

$$\{\{\mathbf{v}\}\}=\frac{\mathbf{v}^{int(K)}+\mathbf{v}^{ext(K)}}{2},\quad [\![\mathbf{v}]\!]_T=\mathbf{n}_K\times\mathbf{v}^{int(K)}-\mathbf{n}_K\times\mathbf{v}^{ext(K)}.$$

For any boundary face $e\subset\mathcal{E}_\mathcal{D}$ which is belong to element K, we define

$$\mathbf{v}^{int}(\mathbf{x})=\mathbf{v}^{int(K)}(\mathbf{x}),\quad \forall\mathbf{x}\in e.$$

To introduce the DG formulation, we rewrite (2.5a)-(2.5b) and (2.3) in the form

$$\mathbf{Q}\mathbf{u}_t+\nabla\cdot\mathbf{f}(\mathbf{u})+\mathcal{P}\mathbf{u}=\mathbf{F}\qquad \text{in }\ \Omega\times(0,T], \tag{3.1a}$$
$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})\qquad \text{in }\ \Omega, \tag{3.1b}$$

where

$$\mathbf{u}=\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix},\quad \mathbf{u}_0=\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix},\quad \mathbf{Q}=\begin{pmatrix}\epsilon_0 I_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & \mu_0 I_{3\times3}\end{pmatrix},\quad \mathbf{f}(\mathbf{u})=[\mathbf{f}_1(\mathbf{u}),\mathbf{f}_2(\mathbf{u}),\mathbf{f}_3(\mathbf{u})]^T,$$
$$\mathbf{f}_i(\mathbf{u})=\begin{pmatrix}-\mathbf{e}_i\times\mathbf{H}\\ \mathbf{e}_i\times\mathbf{E}\end{pmatrix},\quad \mathcal{P}\mathbf{u}=\begin{pmatrix}\mathcal{J}\mathbf{E}\\ \mathcal{K}\mathbf{H}\end{pmatrix},\quad \mathbf{F}=\begin{pmatrix}-e^{-\Gamma_e t}\mathbf{J}_0(x)\\ -e^{-\Gamma_m t}\mathbf{K}_0(x)\end{pmatrix}.$$

Multiplying (3.1a) by a test function $\mathbf{v}=(v_\mathbf{E},v_\mathbf{H})^T\in\mathbf{V}_h^p$, and then integrating by parts over K, we obtain the following weak formulation,

$$\int_K \mathbf{Q}\mathbf{u}_t\cdot\mathbf{v}d\mathbf{x}+\int_{\partial K}(\mathbf{f}(\mathbf{u})\cdot\mathbf{n}_K)\cdot\mathbf{v}ds-\int_K\mathbf{f}(\mathbf{u})\cdot\nabla\mathbf{v}d\mathbf{x}+\int_K\mathcal{P}\mathbf{u}\cdot\mathbf{v}d\mathbf{x}=\int_K\mathbf{F}\cdot\mathbf{v}d\mathbf{x}. \tag{3.2}$$

Based on it, the corresponding semi-discrete DG Scheme is to find $\mathbf{u}_h\in\mathbf{V}_h^p$ such that

$$\begin{aligned}&\int_K\mathbf{Q}\frac{\partial}{\partial t}\mathbf{u}_h\cdot\mathbf{v}_hd\mathbf{x}+\sum_{e\in K}\int_e\big(\widehat{\mathbf{f}(\mathbf{u}_h)\cdot\mathbf{n}_K}\big)\cdot\mathbf{v}_hds\\&-\int_K\mathbf{f}(\mathbf{u}_h)\cdot\nabla\mathbf{v}_hd\mathbf{x}+\int_K\mathcal{P}\mathbf{u}_h\cdot\mathbf{v}_hd\mathbf{x}=\int_K\mathbf{F}\cdot\mathbf{v}_hd\mathbf{x},\end{aligned}\tag{3.3}$$

for all $\mathbf{v}_h\in\mathbf{V}_h^p$ and all elements K, where $\widehat{\mathbf{f}(\mathbf{u}_h)\cdot\mathbf{n}_K}$ is the numerical flux on the face $e\subset\mathcal{E}$.

Following the strategy in [3,10,25], we take

$$\widehat{\mathbf{f}(\mathbf{u}_h)\cdot\mathbf{n}_K}=\begin{pmatrix}-\mathbf{n}_K\times\Big(\{\{\mathbf{H}_h\}\}+\dfrac{1}{2Z}[\![\mathbf{E}_h]\!]_T\Big)\\ \mathbf{n}_K\times\Big(\{\{\mathbf{E}_h\}\}-\dfrac{Z}{2}[\![\mathbf{H}_h]\!]_T\Big)\end{pmatrix}$$

on an interior face $e=\partial K\bigcap\mathcal{E}_I$ and

$$\widehat{\mathbf{f}(\mathbf{u}_h)\cdot\mathbf{n}}=\begin{pmatrix}-\mathbf{n}\times\Big(\mathbf{H}_h^{int}+\dfrac{1}{Z}\mathbf{n}\times\mathbf{E}_h^{int}\Big)\\ \mathbf{0}_{3\times1}\end{pmatrix}$$

on a boundary face $e=\partial K\bigcap\mathcal{E}_D$, where $Z=\sqrt{\mu_0/\epsilon_0}$ denotes the impedance of the medium. Obviously this numerical flux is consistent with $\mathbf{f}(\mathbf{u})\cdot\mathbf{n}_K$.

Moreover, set

$$\mathbf{u}_h(\mathbf{x},0)=\pi_h\mathbf{u}_0(\mathbf{x}), \tag{3.4}$$

where π_h is the element-wise L^2 projection operator.

Summing up over $K\in\mathcal{T}_h$ in (3.2) and (3.3), we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K\mathbf{Q}\mathbf{u}_t\cdot\mathbf{v}d\mathbf{x}+\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbf{f}(\mathbf{u})\cdot\mathbf{n}_K)\cdot\mathbf{v}ds\\&-\sum_{K\in\mathcal{T}_h}\int_K\mathbf{f}(\mathbf{u})\cdot\nabla\mathbf{v}d\mathbf{x}+\sum_{K\in\mathcal{T}_h}\int_K\mathcal{P}\mathbf{u}\cdot\mathbf{v}d\mathbf{x}=\int_K\mathbf{F}\cdot\mathbf{v}d\mathbf{x},\end{aligned}\tag{3.5}$$

and

$$\sum_{K\in\mathcal{T}_h}\int_K \mathbf{Q}\frac{\partial}{\partial t}\mathbf{u}_h\cdot\mathbf{v}_h d\mathbf{x}+\sum_{K\in\mathcal{T}_h}\sum_{e\in K}\int_e \big(\widehat{\mathbf{f}(\mathbf{u}_h)\cdot\mathbf{n}_K}\big)\cdot\mathbf{v}_h ds$$
$$-\sum_{K\in\mathcal{T}_h}\int_K \mathbf{f}(\mathbf{u}_h)\cdot\nabla\mathbf{v}_h d\mathbf{x}+\sum_{K\in\mathcal{T}_h}\int_K \mathcal{P}\mathbf{u}_h\cdot\mathbf{v}_h d\mathbf{x}=\sum_{K\in\mathcal{T}_h}\int_K \mathbf{F}\cdot\mathbf{v}_h d\mathbf{x}. \tag{3.6}$$

Before formulating the fully discrete scheme, we need to approximate the integral terms. Recalling the definition of $\mathcal{J}\mathbf{E}$, we have

$$\begin{aligned}\mathcal{J}^k\mathbf{E}\triangleq\mathcal{J}\mathbf{E}\big|_{t=t_k}&=\epsilon_0\omega_{pe}^2\int_0^{t_k}\mathbf{E}(\mathbf{x},s)e^{-\Gamma_e(t_k-s)}ds\\&=\epsilon_0\omega_{pe}^2\Big(\int_0^{t_{k-1}}+\int_{t_{k-1}}^{t_k}\Big)\mathbf{E}(\mathbf{x},s)e^{-\Gamma_e(t_k-s)}ds\\&=e^{-\Gamma_e\tau}\mathcal{J}^{k-1}\mathbf{E}+\epsilon_0\omega_{pe}^2\int_{t_{k-1}}^{t_k}\mathbf{E}(\mathbf{x},s)e^{-\Gamma_e(t_k-s)}ds\\&\approx e^{-\Gamma_e\tau}\mathcal{J}^{k-1}\mathbf{E}+\frac{\tau}{2}\epsilon_0\omega_{pe}^2(\mathbf{E}^k+e^{-\Gamma_e\tau}\mathbf{E}^{k-1}).\end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned}\mathcal{K}^k\mathbf{H}\triangleq\mathcal{K}\mathbf{H}\big|_{t=t_k}&=\mu_0\omega_{pm}^2\int_0^{t_k}\mathbf{H}(\mathbf{x},s)e^{-\Gamma_m(t_k-s)}ds\\&=\epsilon_0\omega_{pm}^2\Big(\int_0^{t_{k-1}}+\int_{t_{k-1}}^{t_k}\Big)\mathbf{H}(s)e^{-\Gamma_m(t_k-s)}ds\\&=e^{-\Gamma_m\tau}\mathcal{K}^{k-1}\mathbf{E}+\mu_0\omega_{pm}^2\int_{t_{k-1}}^{t_k}\mathbf{H}(\mathbf{x},s)e^{-\Gamma_m(t_k-s)}ds\\&\approx e^{-\Gamma_m\tau}\mathcal{K}^{k-1}\mathbf{H}+\frac{\tau}{2}\mu_0\omega_{pm}^2(\mathbf{H}^k+e^{-\Gamma_m\tau}\mathbf{H}^{k-1}).\end{aligned} \tag{3.8}$$

As a result, $\mathcal{J}\mathbf{E}_h\big|_{t=t_k}$ and $\mathcal{K}\mathbf{H}_h\big|_{t=t_k}$ will be approximated by $\mathcal{J}^k\mathbf{E}_h$ and $\mathcal{K}^k\mathbf{H}_h$, which are expressed as

$$\mathcal{J}^k\mathbf{E}_h=e^{-\Gamma_e\tau}\mathcal{J}^{k-1}\mathbf{E}_h+\frac{\tau}{2}\epsilon_0\omega_{pe}^2(\mathbf{E}_h^k+e^{-\Gamma_e\tau}\mathbf{E}_h^{k-1})\qquad\text{for }\ 1\le k\le N, \tag{3.9a}$$
$$\mathcal{K}^k\mathbf{H}_h=e^{-\Gamma_m\tau}\mathcal{K}^{k-1}\mathbf{H}_h+\frac{\tau}{2}\mu_0\omega_{pm}^2(\mathbf{H}_h^k+e^{-\Gamma_m\tau}\mathbf{H}_h^{k-1})\qquad\text{for }\ 1\le k\le N, \tag{3.9b}$$

with $\mathbf{J}^0\mathbf{E}_h=0$, $\mathbf{K}^0\mathbf{H}_h=0$.

In a similar way, $\mathcal{J}^k\mathbf{E}$ and $\mathcal{K}^k\mathbf{H}$ will be approximated by $\mathcal{J}_a^k\mathbf{E}$ and $\mathcal{K}_a^k\mathbf{H}$, which are expressed as

$$\mathcal{J}_a^k\mathbf{E}=e^{-\Gamma_e\tau}\mathcal{J}_a^{k-1}\mathbf{E}+\frac{\tau}{2}\epsilon_0\omega_{pe}^2(\mathbf{E}^k+e^{-\Gamma_e\tau}\mathbf{E}^{k-1})\qquad\text{for }\ 1\le k\le N, \tag{3.10a}$$
$$\mathcal{K}_a^k\mathbf{H}_h=e^{-\Gamma_m\tau}\mathcal{K}_a^{k-1}\mathbf{H}+\frac{\tau}{2}\mu_0\omega_{pm}^2(\mathbf{H}^k+e^{-\Gamma_m\tau}\mathbf{H}^{k-1})\qquad\text{for }\ 1\le k\le N, \tag{3.10b}$$

with $\mathcal{J}_a^0\mathbf{E}=0$, $\mathcal{K}_a^0\mathbf{H}=0$.

Actually the approximation in (3.9a) and (3.10a) are based on the compound trapezoid formula, whose approximation error is of second order.

On the basis of the semi-discrete scheme (3.6), we introduce a Crank-Nicolson scheme for temporal discretization to obtain the following fully discrete scheme: for any $k\in\{0,1,2,\cdots,N-1\}$, find $\mathbf{u}_h^{k+1}=(\mathbf{E}_h^{k+1},\mathbf{H}_h^{k+1})^T$ such that

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K\mathbf{Q}\delta_t\mathbf{u}_h^{k+1}\cdot\mathbf{v}_h d\mathbf{x}+\sum_{K\in\mathcal{T}_h}\sum_{e\in K}\int_{\partial K}\Big(\widehat{\mathbf{f}(\bar{\mathbf{u}}_h^{k+1})\cdot\mathbf{n}_K}\Big)\cdot\mathbf{v}_h ds-\sum_{K\in\mathcal{T}_h}\int_K\mathbf{f}(\bar{\mathbf{u}}_h^{k+1})\cdot\nabla\mathbf{v}_h d\mathbf{x}\\&+\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K(\mathcal{P}^{k+1}\mathbf{u}_h+\mathcal{P}^k\mathbf{u}_h)\cdot\mathbf{v}_h d\mathbf{x}=\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K(\mathbf{F}_h^k+\mathbf{F}_h^{k+1})\cdot\mathbf{v}_h d\mathbf{x},\end{aligned}\tag{3.11}$$

where

$$\bar{u}^{k+1}=\frac{1}{2}(u^k+u^{k+1}),\qquad \delta_t u_h^{k+1}=\frac{1}{\tau}(u^{k+1}-u^k),$$

$$\mathcal{P}^k\mathbf{u}_h=\begin{pmatrix}\mathcal{J}^k\mathbf{E}_h\\ \mathcal{K}^k\mathbf{H}_h\end{pmatrix},\qquad \mathbf{F}_h^k=\begin{pmatrix}-e^{-\Gamma_e t_k}\pi_h\mathcal{J}_0(\mathbf{x})\\ -e^{-\Gamma_m t_k}\pi_h\mathcal{K}_0(\mathbf{x})\end{pmatrix}.$$

4 L^2 stability analysis and error analysis

4.1 The L^2 stability

In this section, we will carry out the stability analysis of the numerical scheme. First, we define a bilinear form as

$$B(\mathbf{u}_h,\mathbf{v}_h)=\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\int_e\Big(\widehat{\mathbf{f}(\mathbf{u}_h)\cdot\mathbf{n}_K}\Big)\cdot\mathbf{v}_h ds-\sum_{K\in\mathcal{T}_h}\int_K\mathbf{f}(\mathbf{u}_h)\cdot\nabla\mathbf{v}_h(\mathbf{x},t)d\mathbf{x}.\tag{4.1}$$

To prove the stability of the scheme (3.11), we need the following lemmas.

Lemma 4.1 (see [13])**.** *For any $k\geq1$, we have*

$$\|\mathcal{J}^k\mathbf{E}_h\|_0\leq\tau\epsilon_0\omega_{pe}^2\sum_{i=0}^k\|\mathbf{E}_h^i\|_0,\quad \|\mathcal{K}^k\mathbf{H}_h\|_0\leq\tau\mu_0\omega_{pm}^2\sum_{i=0}^k\|\mathbf{H}_h^i\|_0.$$

Lemma 4.2 (The discrete Gronwall inequality)**.** *Let $f(t)$ and $g(t)$ be nonnegative functions defined on $t_k=k\tau$, $k=0,1,\cdots,N,N\tau=T$, and $g(t)$ be non-decreasing. If*

$$f(t_k)\leq g(t_k)+C\tau\sum_{j=0}^{k-1}f(t_j),$$

where C is a positive constant, then

$$f(t_k)\leq g(t_k)e^{Ck\tau}.$$

Lemma 4.3 (see [25]). *For any $\mathbf{u}_h(\cdot,t)\in\mathbf{V}_h^p$, $t\in(0,T)$, we have*

$$B(\mathbf{u}_h,\mathbf{u}_h)=\sum_{e\in\mathcal{E}_\mathcal{I}}\int_e\frac{1}{Z}\big|[\![\mathbf{E}_h(\mathbf{x},t)]\!]_T\big|^2+Z\big|[\![\mathbf{H}_h(\mathbf{x},t)]\!]_T\big|^2ds+2\sum_{e\in\mathcal{E}_\mathcal{D}}\int_e\frac{1}{Z}\big|\mathbf{n}\times\mathbf{E}_h^{int}\big|^2ds.$$

At this point, we give the stability of the DG scheme (3.11).

Theorem 4.1. *For any $n\ge1$, we have*

$$\|\mathbf{Q}^{\frac12}\mathbf{u}_h^n\|_0^2\le C(\|\mathbf{Q}^{\frac12}\mathbf{u}_h^0\|_0^2+\|\mathbf{J}_0\|_0^2+\|\mathbf{K}_0\|_0^2),$$

where C is a genetic constant, which is only dependent on T, but independent of both the mesh size h and time step τ.

Proof. Choosing $\mathbf{v}_h=2\tau\bar{\mathbf{u}}_h^{k+1}$ in (3.11), we get

$$\begin{aligned}&\|\mathbf{Q}^{\frac12}\mathbf{u}_h^{k+1}\|^2-\|\mathbf{Q}^{\frac12}\mathbf{u}_h^{k}\|^2+2\tau B(\bar{\mathbf{u}}_h^{k+1},\bar{\mathbf{u}}_h^{k+1})+\tau(\mathcal{P}^{k+1}\mathbf{u}_h+\mathcal{P}^{k}\mathbf{u}_h,\bar{\mathbf{u}}_h^{k+1})_\Omega\\=&\tau(\mathbf{F}_h^k+\mathbf{F}_h^{k+1},\bar{\mathbf{u}}_h^{k+1})_\Omega.\end{aligned}$$

According to the definition of $\mathcal{P}^k\mathbf{u}_h$ and (3.9a), a straightforward calculation leads to

$$\begin{aligned}&\|\mathbf{Q}^{\frac12}\mathbf{u}_h^{k+1}\|^2-\|\mathbf{Q}^{\frac12}\mathbf{u}_h^{k}\|^2+2\tau B(\bar{\mathbf{u}}_h^{k+1},\bar{\mathbf{u}}_h^{k+1})+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2(\mathbf{E}_h^{k+1},\mathbf{E}_h^{k+1})\\&+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2e^{-\Gamma_e\tau}(\mathbf{E}_h^{k},\mathbf{E}_h^{k})+\frac{\tau^2}{4}\mu_0\omega_{pm}^2(\mathbf{H}_h^{k+1},\mathbf{H}_h^{k+1})+\frac{\tau^2}{4}\mu_0\omega_{pm}^2e^{-\Gamma_m\tau}(\mathbf{H}_h^{k},\mathbf{H}_h^{k})\\=&-\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2(1+e^{-\Gamma_e\tau})(\mathbf{E}_h^{k+1},\mathbf{E}_h^{k})-\frac{\tau^2}{4}\mu_0\omega_{pm}^2(1+e^{-\Gamma_e\tau})(\mathbf{H}_h^{k+1},\mathbf{H}_h^{k})\\&+\tau(\mathbf{F}_h^k+\mathbf{F}_h^{k+1},\bar{\mathbf{u}}_h^{k+1})-\tau(1+e^{-\Gamma_e\tau})(\mathcal{J}^k\mathbf{E}_h,\bar{\mathbf{E}}_h^{k+1})-\tau(1+e^{-\Gamma_m\tau})(\mathcal{K}^k\mathbf{H}_h,\bar{\mathbf{H}}_h^{k+1}).\end{aligned}\tag{4.2}$$

By using the Cauchy-Schwarz inequality and the Young's inequality, we get

$$(\mathbf{E}_h^{k+1},\mathbf{E}_h^{k})\le\delta_1\|\mathbf{E}_h^{k+1}\|_0^2+\frac{1}{4\delta_1}\|\mathbf{E}_h^{k}\|_0^2,\tag{4.3a}$$

$$(\mathbf{H}_h^{k+1},\mathbf{H}_h^{k})\le\delta_1\|\mathbf{H}_h^{k+1}\|_0^2+\frac{1}{4\delta_1}\|\mathbf{H}_h^{k}\|_0^2,\tag{4.3b}$$

and

$$\tau\Big(\frac{e^{-\Gamma_et_k}+e^{-\Gamma_et_{k+1}}}{2}\Big)(\mathbf{J}_0,\mathbf{E}_h^{k+1}+\mathbf{E}_h^{k})_\Omega\le\tau\delta_2(\|\mathbf{E}_h^{k+1}\|_0^2+\|\mathbf{E}_h^{k}\|_0^2)+\frac{\tau}{2\delta_2}\|\mathbf{J}_0\|_0^2,\tag{4.4a}$$

$$\tau\Big(\frac{e^{-\Gamma_mt_k}+e^{-\Gamma_mt_{k+1}}}{2}\Big)(\mathbf{K}_0,\mathbf{H}_h^{k+1}+\mathbf{H}_h^{k})_\Omega\le\tau\delta_3(\|\mathbf{H}_h^{k+1}\|_0^2+\|\mathbf{H}_h^{k}\|_0^2)+\frac{\tau}{2\delta_3}\|\mathbf{K}_0\|_0^2.\tag{4.4b}$$

According to Lemma 4.1,

$$\|\mathcal{J}^k\mathbf{E}_h\|_0^2 \le \tau^2(\epsilon_0\omega_{pe}^2)^2(k+1)\sum_{i=0}^{k}\|\mathbf{E}_h^i\|_0^2 \le C(T)\tau\sum_{i=0}^{k}\|\mathbf{E}_h^i\|_0^2,$$
$$\|\mathcal{K}^k\mathbf{H}_h\|_0^2 \le \tau^2(\mu_0\omega_{pm}^2)^2(k+1)\sum_{i=0}^{k}\|\mathbf{H}_h^i\|_0^2 \le C(T)\tau\sum_{i=0}^{k}\|\mathbf{H}_h^i\|_0^2.$$

Then, by the Cauchy-Schwarz inequality and the Young's inequality again, we have

$$\begin{aligned}\tau\Big((1+e^{-\Gamma_e\tau})\mathcal{J}^k\mathbf{E}_h,\bar{\mathbf{E}}_h^{k+1}\Big) &\le \frac{\tau}{2}\|(1+e^{-\Gamma_e\tau})\mathcal{J}^k\mathbf{E}_h\|_0\|\mathbf{E}_h^{k+1}+\mathbf{E}_h^k\|_0\\ &\le \delta_2\tau(\|\mathbf{E}_h^{k+1}\|_0^2+\|\mathbf{E}_h^k\|_0^2)+\frac{C(T)\tau^2}{2\delta_2}\sum_{i=0}^{k}\|\mathbf{E}_h^i\|_0^2. \end{aligned}\tag{4.5}$$

Similarly,

$$\tau\Big((1+e^{-\Gamma_m\tau})\mathcal{K}^k\mathbf{H}_h,\bar{\mathbf{H}}_h^{k+1}\Big) \le \delta_3\tau(\|\mathbf{H}_h^{k+1}\|_0^2+\|\mathbf{H}_h^k\|_0^2)+\frac{C(T)\tau^2}{2\delta_3}\sum_{i=0}^{k}\|\mathbf{H}_h^i\|_0^2. \tag{4.6}$$

Substituting (4.3a), (4.4a), (4.5) and (4.6) into (4.2), we have

$$\begin{aligned}&\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^{k+1}\|^2-\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^k\|^2+2\tau B(\bar{\mathbf{u}}^{k+1},\bar{\mathbf{u}}^{k+1})+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2\|\mathbf{E}_h^{k+1}\|_0^2\\ &+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2e^{-\Gamma_e\tau}\|\mathbf{E}_h^k\|_0^2+\frac{\tau^2}{4}\mu_0\omega_{pm}^2\|\mathbf{H}_h^{k+1}\|_0^2+\frac{\tau^2}{4}\mu_0\omega_{pm}^2e^{-\Gamma_m\tau}\|\mathbf{E}_h^{k+1}\|_0^2\\ \le&\frac{\tau^2}{2}\epsilon_0\omega_{pe}^2\Big(\delta_1\|\mathbf{E}_h^{k+1}\|^2+\frac{1}{4\delta_1}\|\mathbf{E}_h^k\|^2\Big)+\frac{\tau^2}{2}\mu_0\omega_{pm}^2\Big(\delta_1\|\mathbf{H}_h^{k+1}\|^2+\frac{1}{4\delta_1}\|\mathbf{H}_h^k\|^2\Big)\\ &+2\tau\delta_2(\|\mathbf{E}_h^{k+1}\|^2+\|\mathbf{E}_h^k\|^2)+\frac{\tau}{2\delta_2}\|\mathbf{J}_0\|^2+2\tau\delta_3(\|\mathbf{H}_h^{k+1}\|^2+\|\mathbf{H}_h^k\|^2)+\frac{\tau}{2\delta_3}\|\mathbf{K}_0\|^2\\ &+\frac{C(T)\tau^2}{2\delta_2}\sum_{i=0}^{k}\|\mathbf{E}_h^i\|^2+\frac{C(T)\tau^2}{2\delta_3}\sum_{i=0}^{k}\|\mathbf{H}_h^i\|^2.\end{aligned}$$

Summing up the inequality above from $k=0$ to $k=n-1$, using the fact that $B(\bar{\mathbf{u}}_h^{k+1},\bar{\mathbf{u}}_h^{k+1})\ge 0$, and assuming that $\tau\le 1$ for simplicity, we obtain

$$\begin{aligned}&\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^n\|^2-\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^0\|^2+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\sum_{k=1}^{n}\|\mathbf{E}_h^k\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\sum_{k=1}^{n}\|\mathbf{H}_h^k\|_0^2\\ \le&\frac{\tau^2\epsilon_0\omega_{pe}^2\delta_1}{2}\sum_{k=1}^{n}\|\mathbf{E}_h^k\|^2+\frac{\tau^2\mu_0\omega_{pm}^2\delta_1}{2}\sum_{k=1}^{n}\|\mathbf{H}_h^k\|^2+\frac{\tau^2\epsilon_0\omega_{pe}^2}{8\delta_1}\sum_{k=0}^{n-1}\|\mathbf{E}_h^k\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{8\delta_1}\sum_{k=0}^{n-1}\|\mathbf{H}_h^k\|_0^2\\ &+4\delta_2\tau\sum_{k=0}^{n-1}\|\mathbf{E}_h^k\|^2+\frac{\tau n}{2\delta_2}\|\mathbf{J}_0\|^2+4\delta_3\tau\sum_{k=0}^{n-1}\|\mathbf{H}_h^k\|^2+\frac{\tau n}{2\delta_3}\|\mathbf{K}_0\|^2+2\delta_2\|\mathbf{E}_h^n\|^2+2\delta_3\|\mathbf{H}_h^n\|^2\\ &+\frac{C(T)\tau^2 n}{2\delta_2}\sum_{k=0}^{n-1}\|\mathbf{E}_h^k\|^2+\frac{C(T)\tau^2 n}{2\delta_3}\sum_{k=0}^{n-1}\|\mathbf{H}_h^k\|^2.\end{aligned}$$

Choosing $\delta_1=1/2$, $\delta_2=\epsilon_0/4$ and $\delta_3=\mu_0/4$, we get

$$\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^n\|^2-\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^0\|^2\le C(T)\|\mathbf{J}_0\|^2+C(T)\|\mathbf{K}_0\|^2+C(T)\tau\sum_{k=0}^{n-1}\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_h^k\|^2.$$

By Lemma 4.2, we conclude the proof. □

4.2 L^2 error estimate

Now we turn to the error estimate of our approach. We first introduce the following lemmas.

Lemma 4.4 (see [17]). *Denote $\mathbf{u}^k=\mathbf{u}(\cdot,k\tau)$. For any $\mathbf{u}\in H^2([0,T],(L^2(\Omega))^3)$, we have*

$$\Big\|\tau\bar{\mathbf{u}}^{k+1}-\int_{I_k}\mathbf{u}(s)ds\Big\|_0^2\le\frac{\tau^5}{4}\int_{I_k}\|\mathbf{u}_{tt}(s)\|^2ds.$$

Moreover, we also need the element-wise L^2-projection operator $\pi_h: H^{p+1}(\Omega)\to\mathbf{V}_h^p$, which satisfies

$$\int_K\big(u-\pi_hu\big)vd\mathbf{x}=0,\quad\forall v\in P^p(K),\quad\forall K\in\mathcal{T}_h.$$

For this L^2 projection operator, we have the following approximation lemma.

Lemma 4.5. *Let $u\in H^{p+1}(K)$. Then*

$$\|u-\pi_hu\|_{0,K}\le Ch^{p+1}|u|_{p+1,K},\quad\|u-\pi_hu\|_{0,\partial K}\le Ch^{p+1/2}|u|_{p+1,K}.$$

Take $\mathbf{R}_\mathbf{u}^k=\mathbf{u}^k-\pi_h\mathbf{u}^k$, $\theta_\mathbf{u}^k=\mathbf{u}_h^k-\pi_h\mathbf{u}^k$. Denote $\mathbf{e}_\mathbf{u}^k=\mathbf{u}^k-\mathbf{u}_h^k$. Thus, $\mathbf{e}_\mathbf{u}^k=\mathbf{R}_\mathbf{u}^k-\theta_\mathbf{u}^k$. By the definition of the L^2 projection π_h, we have the following orthogonal property,

$$(\mathbf{R}_\mathbf{u}^i,\theta_\mathbf{u}^j)=0\quad\text{for all }\ 0\le i,j\le N. \tag{4.7}$$

Lemma 4.6. *Let $\mathcal{J}^k\mathbf{E}$ be defined in* (3.7)*. Then for any $\mathbf{E}\in H([0,T],(L^2(\Omega))^3)$,*

$$\begin{aligned}&\Big|\int_{t_k}^{t_{k+1}}(\mathcal{J}\mathbf{E},\bar{\theta}_\mathbf{E}^{k+1})dt-\frac{\tau}{2}(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^k\mathbf{E},\bar{\theta}_\mathbf{E}^{k+1})\Big|\\ \le&\frac{\tau\epsilon_0}{20}(\|\theta_\mathbf{E}^{k+1}\|_0^2+\|\theta_\mathbf{E}^k\|_0^2)+C\tau^4\epsilon_0\int_{t_k}^{t_{k+1}}\|\mathbf{E}_t\|_0^2dt.\end{aligned}$$

Proof. By Lemma 4.4, we have

$$\begin{aligned}&\Big|\int_{t_k}^{t_{k+1}}(\mathcal{J}\mathbf{E},\bar{\theta}_\mathbf{E}^{k+1})dt-\frac{\tau}{2}(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^k\mathbf{E},\bar{\theta}_\mathbf{E}^{k+1})\Big|\\ \le&\Big\|\int_{t_k}^{t_{k+1}}\mathcal{J}\mathbf{E}dt-\frac{\tau}{2}(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^k\mathbf{E})\Big\|_0\|\bar{\theta}_\mathbf{E}^{k+1}\|_0\\ \le&\frac{\tau\epsilon_0}{10}\|\bar{\theta}_\mathbf{E}^{k+1}\|^2+C\tau^4\epsilon_0\int_{t_k}^{t_{k+1}}\|(\mathcal{J}\mathbf{E})_{tt}\|^2dt\\ \le&\frac{\tau\epsilon_0}{20}(\|\theta_\mathbf{E}^{k+1}\|_0^2+\|\theta_\mathbf{E}^k\|_0^2)+C\tau^4\epsilon_0\int_{t_k}^{t_{k+1}}\|\mathbf{E}_t\|^2dt.\end{aligned}$$

So, we complete the proof. □

Lemma 4.7. *Assume* $\mathbf{E}\in L^2([0,T],(L^2(\Omega))^3)$. *Let* $\mathcal{J}^k\mathbf{E}_h$ *be defined by* (3.9a) *and* $\mathcal{J}_a^k\mathbf{E}$ *defined by* (3.10a). *Then for any* $1\le k\le N$,

$$(\mathcal{J}^k\mathbf{E}_h-\mathcal{J}_a^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\le\frac{\epsilon_0}{20}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^k\|_0^2)+C(T)\tau\sum_{j=0}^{k}\epsilon_0\|\theta_{\mathbf{E}}^j\|_0^2.$$

Proof. By the definition of (3.9a) and (3.10a), and the induction argument, a direct calculation leads to

$$\begin{aligned}
&\mathcal{J}^k\mathbf{E}_h-\mathcal{J}_a^k\mathbf{E}\\
=&e^{-\Gamma_e\tau}(\mathcal{J}^{k-1}\mathbf{E}_h-\mathcal{J}_a^{k-1}\mathbf{E})+\frac{\tau\epsilon_0\omega_{pe}^2}{2}\big(\mathbf{E}_h^k-\mathbf{E}^k+e^{-\Gamma_e\tau}(\mathbf{E}_h^{k-1}-\mathbf{E}_h^{k-1})\big)\\
=&e^{-\Gamma_e\tau}\Big(e^{-\Gamma_e\tau}(\mathcal{J}^{k-2}\mathbf{E}_h-\mathcal{J}_a^{k-2}\mathbf{E})+\frac{\tau\epsilon_0\omega_{pe}^2}{2}\big(\mathbf{E}_h^{k-1}-\mathbf{E}^{k-1}+e^{-\Gamma_e\tau}(\mathbf{E}_h^{k-2}-\mathbf{E}_h^{k-2})\big)\Big)\\
&+\frac{\tau\epsilon_0\omega_{pe}^2}{2}\big(\mathbf{E}_h^k-\mathbf{E}^k+e^{-\Gamma_e\tau}(\mathbf{E}_h^{k-1}-\mathbf{E}_h^{k-1})\big)\\
=&\frac{\tau\epsilon_0\omega_{pe}^2}{2}\sum_{j=0}^{k-1}e^{-j\Gamma_e\tau}\big(\mathbf{E}_h^{k-j}-\mathbf{E}^{k-j}+e^{-\Gamma_e\tau}(\mathbf{E}_h^{k-j-1}-\mathbf{E}^{k-j-1})\big).
\end{aligned}$$

The combination of the Cauchy-Schwarz inequality, the Young's inequality and (4.7) yields

$$\begin{aligned}
&(\mathcal{J}^k\mathbf{E}_h-\mathcal{J}_a^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\\
=&\Big(\frac{\tau\epsilon_0\omega_{pe}^2}{2}\sum_{j=0}^{k-1}e^{-j\Gamma_e\tau}\Big(\mathbf{E}_h^{k-j}-\mathbf{E}^{k-j}+e^{-\Gamma_e\tau}(\mathbf{E}_h^{k-j-1}-\mathbf{E}^{k-j-1})\Big),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\
=&\Big(\frac{\tau\epsilon_0\omega_{pe}^2}{2}\sum_{j=0}^{k-1}e^{-j\Gamma_e\tau}(\theta_{\mathbf{E}}^{k-j}+e^{-\Gamma_e\tau}\theta_{\mathbf{E}}^{k-j-1}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\
\le&\frac{\epsilon_0}{20}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^k\|_0^2)+C(T)\tau\sum_{j=0}^{k}\epsilon_0\|\theta_{\mathbf{E}}^j\|_0^2.
\end{aligned}$$

Thus, the proof is completed. □

By Lemma 4.4, we can obtain

Lemma 4.8. *For any* $\mathbf{E}\in H^2([0,T],(L^2(\Omega))^3)$,

$$\Big\|\int_{t_k}^{t_{k+1}}e^{-\Gamma_e(t_{k+1}-s)}\mathbf{E}(s)ds-\frac{\tau}{2}(\mathbf{E}^{k+1}+e^{-\Gamma_e\tau}\mathbf{E}^k)\Big\|_0^2\le\frac{C\tau^5}{4}\int_{t_k}^{t_{k+1}}(\|\mathbf{E}_{tt}\|_0^2+\|\mathbf{E}_t\|_0^2+\|\mathbf{E}\|_0^2)ds.$$

Lemma 4.9. *Let* $\mathcal{J}^k\mathbf{E}$ *be defined by* (3.7) *and* $\mathcal{J}_a^k\mathbf{E}$ *defined by* (3.10a). *Then for any* $1\le k\le N$, $\mathbf{E}\in H^2([0,T],(L^2(\Omega))^3)$, *we have*

$$(\mathcal{J}_a^k\mathbf{E}-\mathcal{J}^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\le\frac{\epsilon_0}{20}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^k\|_0^2)+C\epsilon_0\tau^5\int_0^{t_k}(\|\mathbf{E}_{tt}\|_0^2+\|\mathbf{E}_t\|_0^2+\|\mathbf{E}\|_0^2)ds.$$

Proof. According to the definition of (3.7) and (3.10a),

$$\begin{aligned}&\mathcal{J}_a^k\mathbf{E}-\mathcal{J}^k\mathbf{E}\\=&e^{-\Gamma_e\tau}(\mathcal{J}_a^{k-1}\mathbf{E}-\mathcal{J}^{k-1}\mathbf{E})+\epsilon_0\omega_{pe}^2\Big(\frac{\tau}{2}(\mathbf{E}^k+e^{-\Gamma_e\tau}\mathbf{E}^{k-1})-\int_{t_{k-1}}^{t_k}e^{-\Gamma_e(t_k-s)}\mathbf{E}(\mathbf{x},s)ds\Big).\end{aligned}$$

By the induction argument, we obtain

$$\mathcal{J}_a^k\mathbf{E}-\mathcal{J}^k\mathbf{E}=\epsilon_0\omega_{pe}^2\sum_{j=0}^{k-1}e^{-j\Gamma_e\tau}\Big(\frac{\tau}{2}(\mathbf{E}^{k-j}+e^{-\Gamma_e\tau}\mathbf{E}^{k-j-1})-\int_{t_{k-j-1}}^{t_{k-j}}e^{-\Gamma_e(t_{k-j}-s)}\mathbf{E}(\mathbf{x},s)ds\Big).$$

By the Cauchy-Schwarz inequality, the Young's inequality and Lemma 4.8, we get

$$\begin{aligned}&(\mathcal{J}_a^k\mathbf{E}-\mathcal{J}^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\\=&\Big(\epsilon_0\omega_{pe}^2\sum_{j=0}^{k-1}e^{-j\Gamma_e\tau}\Big(\frac{\tau}{2}(\mathbf{E}^{k-j}+e^{-\Gamma_e\tau}\mathbf{E}^{k-j-1})-\int_{t_{k-j-1}}^{t_{k-j}}e^{-\Gamma_e(t_{k-j}-s)}\mathbf{E}(\mathbf{x},s)ds\Big),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\\le&\epsilon_0\omega_{pe}^2\sum_{j=0}^{k-1}\Big\|\Big(\frac{\tau}{2}(\mathbf{E}^{k-j}+e^{-\Gamma_e\tau}\mathbf{E}^{k-j-1})-\int_{t_{k-j-1}}^{t_{k-j}}e^{-\Gamma_e(t_{k-j}-s)}\mathbf{E}(\mathbf{x},s)ds\Big)\Big\|_0\cdot\|\bar{\theta}_{\mathbf{E}}^{k+1}\|_0\\\le&\frac{\epsilon_0}{20}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^k\|_0^2)+C\epsilon_0\tau^5\int_0^{t_k}(\|\mathbf{E}_{tt}\|_0^2+\|\mathbf{E}_t\|_0^2+\|\mathbf{E}\|_0^2)ds.\end{aligned}$$

So, we complete the proof. □

Taking $v=v_h$ in (3.5), integrating (3.5) over $I_k=[t_k,t_{k+1}]$ with respect to t on both sides of it and dividing by τ, then subtracting (3.11), we obtain the following error equations

$$\begin{aligned}&\frac{1}{\tau}(\mathbf{Qe}_{\mathbf{u}}^{k+1}-\mathbf{Qe}_{\mathbf{u}}^k,\mathbf{v}_h)+\frac{1}{\tau}\int_{t_k}^{t_{k+1}}B(\mathbf{u},\mathbf{v}_h)dt-B(\bar{\mathbf{u}}_h^{k+1},\mathbf{v}_h)+\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(\mathcal{P}\mathbf{u},\mathbf{v}_h)dt\\&-\frac{1}{2}(\mathcal{P}^{k+1}\mathbf{u}_h+\mathcal{P}^k\mathbf{u}_h,\mathbf{v}_h)=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(\mathbf{F},\mathbf{v}_h)dt-\frac{1}{2}(\mathbf{F}_h^{k+1}+\mathbf{F}_h^k,\mathbf{v}_h).\end{aligned}$$

Which can be rewritten as

$$\begin{aligned}&\frac{1}{\tau}(\mathbf{Qe}_{\mathbf{u}}^{k+1}-\mathbf{Qe}_{\mathbf{u}}^k,\mathbf{v}_h)+\frac{1}{\tau}\int_{t_k}^{t_{k+1}}B(\mathbf{u},\mathbf{v}_h)dt-B(\bar{\mathbf{u}}^{k+1},\mathbf{v}_h)+B(\bar{\mathbf{e}}_{\mathbf{u}}^{k+1},\mathbf{v}_h)\\&+\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(\mathcal{P}\mathbf{u},\mathbf{v}_h)dt-\frac{1}{2}(\mathcal{P}^{k+1}\mathbf{u}+\mathcal{P}^k\mathbf{u},\mathbf{v}_h)+\frac{1}{2}(\mathcal{P}^{k+1}\mathbf{u}+\mathcal{P}^k\mathbf{u}-\mathcal{P}^{k+1}\mathbf{u}_h-\mathcal{P}^k\mathbf{u}_h,\mathbf{v}_h)\\=&\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(\mathbf{F},\mathbf{v}_h)dt-\frac{1}{2}(\mathbf{F}_h^{k+1}+\mathbf{F}_h^k,\mathbf{v}_h).\end{aligned}\tag{4.8}$$

Theorem 4.2. *Let* $\mathbf{E}_h^n$, $\mathbf{H}_h^n$ *be the solution of* (3.11) *and* $\mathbf{E}^n$, $\mathbf{H}^n$ *be the solution of* (2.5a) *and* (2.5b) *at time* $t=t_n$. *Under the following regularity assumptions:*

$$\mathbf{E},\mathbf{H}\in H^2\big([0,T],\mathbf{H}^{p+1}(curl,\Omega)\big),$$

there exists a constant C independent of the mesh size h and the time step τ, such that

$$\begin{aligned}&\epsilon_0\|\mathbf{E}^n-\mathbf{E}_h^n\|_0^2+\mu_0\|\mathbf{H}^n-\mathbf{H}_h^n\|_0^2\\ \le& C\tau^4(\|\mathbf{J}_0\|^2+\|\mathbf{K}_0\|^2)+C\tau^4\int_0^{t_n}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)dt\\ &+C\tau^4\int_0^{t_n}(\|\mathbf{Q}^{\frac12}\mathbf{u}\|_0^2+\|\mathbf{Q}^{\frac12}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac12}\mathbf{u}_{tt}\|_0^2)dt+Ch^{2p+1}\int_0^{t_n}\|\mathbf{u}\|_0^2dt.\end{aligned}$$

Proof. Taking $\mathbf{v}_h=2\tau\bar{\theta}_{\mathbf{u}}^{k+1}$ in (4.8) and using the orthogonal property (4.7), we have

$$\begin{aligned}&\|\mathbf{Q}^{\frac12}\theta_{\mathbf{u}}^{k+1}\|_0^2-\|\mathbf{Q}^{\frac12}\theta_{\mathbf{u}}^{k}\|_0^2+\tau\Big((\mathcal{P}^{k+1}\mathbf{u}_h+\mathcal{P}^{k}\mathbf{u}_h)-(\mathcal{P}^{k+1}\mathbf{u}+\mathcal{P}^{k}\mathbf{u}),\bar{\theta}_{\mathbf{u}}^{k+1}\Big)\\ =&2\int_{t_k}^{t_{k+1}}B(\mathbf{u},\bar{\theta}_{\mathbf{u}}^{k+1})dt-2\tau B(\bar{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})+2\tau B(\bar{\mathbf{e}}_{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})\\ &+2\int_{t_k}^{t_{k+1}}(\mathcal{P}\mathbf{u},\bar{\theta}_{\mathbf{u}}^{k+1})dt-\tau(\mathcal{P}^{k+1}\mathbf{u}+\mathcal{P}^{k}\mathbf{u},\bar{\theta}_{\mathbf{u}}^{k+1})+\Big(\tau(\mathbf{F}_h^{k+1}+\mathbf{F}_h^{k})-2\int_{t_k}^{t_{k+1}}\mathbf{F}dt,\bar{\theta}_{\mathbf{u}}^{k+1}\Big).\end{aligned}$$

Which is equivalent to

$$\begin{aligned}&\|\mathbf{Q}^{\frac12}\theta_{\mathbf{u}}^{k+1}\|_0^2-\|\mathbf{Q}^{\frac12}\theta_{\mathbf{u}}^{k}\|_0^2+\tau\Big((\mathcal{J}^{k+1}\mathbf{E}_h+\mathcal{J}^{k}\mathbf{E}_h)-(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^{k}\mathbf{E}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\ &+\tau\Big((\mathcal{K}^{k+1}\mathbf{H}_h+\mathcal{K}^{k}\mathbf{H}_h)-(\mathcal{K}^{k+1}\mathbf{H}+\mathcal{K}^{k}\mathbf{H}),\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\ =&2\int_{t_k}^{t_{k+1}}B(\mathbf{u},\bar{\theta}_{\mathbf{u}}^{k+1})dt-2\tau B(\bar{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})+2\tau B(\bar{\mathbf{e}}_{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})\\ &+2\int_{t_k}^{t_{k+1}}(\mathcal{J}\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})dt-\tau(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^{k}\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\\ &+2\int_{t_k}^{t_{k+1}}(\mathcal{K}\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})dt-\tau(\mathcal{K}^{k+1}\mathbf{H}+\mathcal{K}^{k}\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})+\Big(\tau(\mathbf{F}_h^{k+1}+\mathbf{F}_h^{k})-2\int_{t_k}^{t_{k+1}}\mathbf{F}dt,\bar{\theta}_{\mathbf{u}}^{k+1}\Big),\end{aligned}\tag{4.9}$$

where the definition of $\mathcal{P}^{k+1}\mathbf{u}_h$, $\mathcal{P}\mathbf{u}$ and $\mathcal{P}^{k+1}\mathbf{u}$ is used.

By the definition of $\mathcal{J}^k\mathbf{E}$, $\mathcal{J}^k\mathbf{E}_h$ and $\mathcal{J}_a^k\mathbf{E}$ defined in (3.7), (3.9a) and (3.10a) respectively, we have

$$\begin{aligned}&\Big((\mathcal{J}^{k+1}\mathbf{E}_h+\mathcal{J}^{k}\mathbf{E}_h)-(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^{k}\mathbf{E}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\ =&\Big((\mathcal{J}^{k+1}\mathbf{E}_h+\mathcal{J}^{k}\mathbf{E}_h)-(\mathcal{J}_a^{k+1}\mathbf{E}+\mathcal{J}_a^{k}\mathbf{E}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)+\Big((\mathcal{J}_a^{k+1}\mathbf{E}+\mathcal{J}_a^{k}\mathbf{E})-(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^{k}\mathbf{E}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\ =&\Big(\frac{\tau}{2}\epsilon_0\omega_{pe}^2(\theta_{\mathbf{E}}^{k+1}+e^{-\Gamma_e\tau}\theta_{\mathbf{E}}^{k}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)+\Big((1+e^{-\Gamma_e\tau})(\mathcal{J}^{k}\mathbf{E}_h-\mathcal{J}_a^{k}\mathbf{E}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\ &+\Big(\frac{\tau\epsilon_0\omega_{pe}^2}{2}(\mathbf{E}^{k+1}+e^{-\Gamma_e\tau}\mathbf{E}^{k})-\epsilon_0\omega_{pe}^2\int_{t_k}^{t_{k+1}}\mathbf{E}(\mathbf{x},s)e^{-\Gamma_e(t_{k+1}-s)}ds,\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\ &+(1+e^{-\Gamma_e\tau})(\mathcal{J}_a^{k}\mathbf{E}-\mathcal{J}^{k}\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1}).\end{aligned}\tag{4.10}$$

Similarly,

$$\begin{aligned}
&\Big((\mathcal{K}^{k+1}\mathbf{H}_h+\mathcal{K}^k\mathbf{H}_h)-(\mathcal{K}^{k+1}\mathbf{H}+\mathcal{K}^k\mathbf{H}),\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\
=&\Big(\frac{\tau}{2}\mu_0\omega_{pm}^2(\theta_{\mathbf{H}}^{k+1}+e^{-\Gamma_m\tau}\theta_{\mathbf{H}}^k),\bar{\theta}_{\mathbf{H}}^{k+1}\Big)+\Big((1+e^{-\Gamma_m\tau})(\mathcal{K}^k\mathbf{H}_h-\mathcal{K}_a^k\mathbf{H}),\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\
&+\Big(\frac{\tau\mu_0\omega_{pm}^2}{2}(\mathbf{H}^{k+1}+e^{-\Gamma_m\tau}\mathbf{H}^k)-\mu_0\omega_{pm}^2\int_{t_k}^{t_{k+1}}\mathbf{H}(\mathbf{x},s)e^{-\Gamma_m(t_{k+1}-s)}ds,\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\
&+(1+e^{-\Gamma_m\tau})(\mathcal{K}_a^k\mathbf{H}-\mathcal{K}^k\mathbf{E},\bar{\theta}_{\mathbf{H}}^{k+1}).
\end{aligned}\tag{4.11}$$

Plugging (4.10) and (4.11) into (4.9) leads to

$$\begin{aligned}
&\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2-\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2(\theta_{\mathbf{E}}^{k+1},\theta_{\mathbf{E}}^{k+1})+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2e^{-\Gamma_e\tau}(\theta_{\mathbf{E}}^{k},\theta_{\mathbf{E}}^{k})\\
&+\frac{\tau^2}{4}\mu_0\omega_{pm}^2(\theta_{\mathbf{H}}^{k+1},\theta_{\mathbf{H}}^{k+1})+\frac{\tau^2}{4}\mu_0\omega_{pe}^2e^{-\Gamma_m\tau}(\theta_{\mathbf{H}}^{k},\theta_{\mathbf{H}}^{k})\\
=&-\tau\Big((1+e^{-\Gamma_e\tau})(\mathcal{J}^k\mathbf{E}_h-\mathcal{J}_a^k\mathbf{E}),\bar{\theta}_{\mathbf{E}}^{k+1}\Big)-\tau\Big((1+e^{-\Gamma_m\tau})(\mathcal{K}^k\mathbf{H}_h-\mathcal{K}_a^k\mathbf{H}),\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\
&-\tau(1+e^{-\Gamma_e\tau})(\mathcal{J}_a^k\mathbf{E}-\mathcal{J}^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})-\tau(1+e^{-\Gamma_m\tau})(\mathcal{K}_a^k\mathbf{H}-\mathcal{K}^k\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})\\
&-\tau\Big(\frac{\tau\epsilon_0\omega_{pe}^2}{2}(\mathbf{E}^{k+1}+e^{-\Gamma_e\tau}\mathbf{E}^k)-\epsilon_0\omega_{pe}^2\int_{t_k}^{t_{k+1}}\mathbf{E}(\mathbf{x},s)e^{-\Gamma_e(t_{k+1}-s)}ds,\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\
&-\tau\Big(\frac{\tau\mu_0\omega_{pm}^2}{2}(\mathbf{H}^{k+1}+e^{-\Gamma_m\tau}\mathbf{H}^k)-\mu_0\omega_{pm}^2\int_{t_k}^{t_{k+1}}\mathbf{H}(\mathbf{x},s)e^{-\Gamma_m(t_{k+1}-s)}ds,\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\
&-\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2(1+e^{-\Gamma_e\tau})(\theta_{\mathbf{E}}^{k+1},\theta_{\mathbf{E}}^{k})-\frac{\tau^2}{4}\mu_0\omega_{pm}^2(1+e^{-\Gamma_m\tau})(\theta_{\mathbf{H}}^{k+1},\theta_{\mathbf{H}}^{k})\\
&+\Big(2\int_{t_k}^{t_{k+1}}B(\mathbf{u},\bar{\theta}_{\mathbf{u}}^{k+1})dt-2\tau B(\bar{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})\Big)\\
&+2\tau B(\bar{\mathbf{e}}_{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})+2\int_{t_k}^{t_{k+1}}(\mathcal{J}\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})dt-\tau(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\\
&+2\int_{t_k}^{t_{k+1}}(\mathcal{K}\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})dt-\tau(\mathcal{K}^{k+1}\mathbf{H}+\mathcal{K}^k\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})+\Big(\tau(\mathbf{F}_h^{k+1}+\mathbf{F}_h^k)-2\int_{t_k}^{t_{k+1}}\mathbf{F}dt,\bar{\theta}_{\mathbf{u}}^{k+1}\Big)\\
\triangleq&\sum_{j=1}^{9}\mathcal{A}_j.
\end{aligned}\tag{4.12}$$

In the following, we just need to estimate all $\mathcal{A}_j$, for $j=1,\cdots,9$, one by one. By Lemma 4.7,

$$\begin{aligned}
\mathcal{A}_1=&-\tau(1+e^{-\Gamma_e\tau})(\mathcal{J}^k\mathbf{E}_h-\mathcal{J}_a^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})-\tau(1+e^{-\Gamma_m\tau})(\mathcal{K}^k\mathbf{H}_h-\mathcal{K}_a^k\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})\\
\le&\frac{\tau\epsilon_0}{10}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^{k}\|_0^2)+C(T)\tau^2\sum_{j=0}^{k}\epsilon_0\|\theta_{\mathbf{E}}^{j}\|_0^2+\frac{\tau\mu_0}{10}(\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\|\theta_{\mathbf{H}}^{k}\|_0^2)
\end{aligned}$$

$$
\begin{aligned}
&+C(T)\tau^2\sum_{j=0}^{k}\mu_0\|\theta_{\mathbf{H}}^j\|_0^2\\
=&\frac{\tau}{10}(\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2)+C(T)\tau^2\sum_{j=0}^{k}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{j}\|_0^2.
\end{aligned}
$$

Due to Lemma 4.9, we have

$$
\begin{aligned}
\mathcal{A}_2=&-\tau(1+e^{-\Gamma_e\tau})(\mathcal{J}_a^k\mathbf{E}-\mathcal{J}^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})-\tau(1+e^{-\Gamma_m\tau})(\mathcal{K}_a^k\mathbf{H}-\mathcal{K}^k\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})\\
\le&\frac{\tau\epsilon_0}{10}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^{k}\|_0^2)+C\tau^6\epsilon_0\int_0^{t_k}(\|\mathbf{E}_{tt}\|_0^2+\|\mathbf{E}_t\|_0^2+\|\mathbf{E}\|_0^2)ds\\
&+\frac{\tau\mu_0}{10}(\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\|\theta_{\mathbf{H}}^{k}\|_0^2)+C\tau^6\mu_0\int_0^{t_k}(\|\mathbf{H}_{tt}\|_0^2+\|\mathbf{H}_t\|_0^2+\|\mathbf{H}\|_0^2)ds\\
=&\frac{\tau}{10}(\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2)+C\tau^6\int_0^{t_k}(\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_{tt}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}\|_0^2)ds.
\end{aligned}
$$

Using Lemma 4.8, the Cauchy-Schwarz inequality and the Young's inequality, we have

$$
\begin{aligned}
\mathcal{A}_3=&-\tau\Big(\frac{\tau\epsilon_0\omega_{pe}^2}{2}(\mathbf{E}^{k+1}+e^{-\Gamma_e\tau}\mathbf{E}^k)-\epsilon_0\omega_{pe}^2\int_{t_k}^{t_{k+1}}\mathbf{E}(\mathbf{x},s)e^{-\Gamma_e(t_{k+1}-s)}ds,\bar{\theta}_{\mathbf{E}}^{k+1}\Big)\\
\le&\frac{\tau\epsilon_0}{10}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^{k}\|_0^2)+C\tau^6\epsilon_0\int_{t_k}^{t_{k+1}}(\|\mathbf{E}_{tt}\|_0^2+\|\mathbf{E}_t\|_0^2+\|\mathbf{E}\|_0^2)ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{A}_4=&-\tau\Big(\frac{\tau\mu_0\omega_{pm}^2}{2}(\mathbf{H}^{k+1}+e^{-\Gamma_m\tau}\mathbf{H}^k)-\mu_0\omega_{pm}^2\int_{t_k}^{t_{k+1}}\mathbf{H}(\mathbf{x},s)e^{-\Gamma_m(t_{k+1}-s)}ds,\bar{\theta}_{\mathbf{H}}^{k+1}\Big)\\
\le&\frac{\tau\mu_0}{10}(\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\|\theta_{\mathbf{H}}^{k}\|_0^2)+C\tau^6\mu_0\int_{t_k}^{t_{k+1}}(\|\mathbf{H}_{tt}\|_0^2+\|\mathbf{H}_t\|_0^2+\|\mathbf{H}\|_0^2)ds.
\end{aligned}
$$

By using the Cauchy-Schwarz inequality and the Young's inequality, we obtain

$$
\begin{aligned}
\mathcal{A}_5=&-\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2(1+e^{-\Gamma_e\tau})(\theta_{\mathbf{E}}^{k+1},\theta_{\mathbf{E}}^{k})-\frac{\tau^2}{4}\mu_0\omega_{pm}^2(1+e^{-\Gamma_m\tau})(\theta_{\mathbf{H}}^{k+1},\theta_{\mathbf{H}}^{k})\\
\le&\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\|\theta_{\mathbf{E}}^{k}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\|\theta_{\mathbf{H}}^{k}\|_0^2.
\end{aligned}
$$

Due to the fact that the numerical flux is consistent, we have $B(\mathbf{u},\mathbf{v}_h)=(\nabla\cdot\mathbf{f}(\mathbf{u}),\mathbf{v}_h)$. By using Lemma 4.4, we get

$$
\begin{aligned}
\mathcal{A}_6=&2\int_{I_k}B(\mathbf{u},\bar{\theta}_{\mathbf{u}}^{k+1})dt-2\tau B(\bar{\mathbf{u}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})\\
\le&\frac{\tau^{\frac{5}{2}}}{2}\Big(\int_{I_k}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)ds\Big)^{\frac{1}{2}}\|\theta_{\mathbf{u}}^{k+1}+\theta_{\mathbf{u}}^{k}\|_0\\
\le&C\tau^4\int_{I_k}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)ds+\frac{\tau}{10}(\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2).
\end{aligned}
$$

Similar to the strategy of the estimate of $B(\mathbf{R},\theta)$ in [25] and the orthogonal property (4.7), a direct calculation yields

$$\begin{aligned}
\mathcal{A}_7 =&\tau B(\bar{e_{\mathbf{u}}}^{k+1},\bar{\theta}_{\mathbf{u}}^{k+1})\\
\leq&\tau\sum_{e\in\mathcal{E}_{\mathcal{I}}}\int_e |\{\{\bar{\mathbf{R}}\}\}_{\mathbf{E}}^{k+1}|^2+|\{\{\bar{\mathbf{R}}\}\}_{\mathbf{H}}^{k+1}|^2+\big(|[\![\bar{\mathbf{R}}_{\mathbf{E}}^{k+1}]\!]_T|^2+|[\![\bar{\mathbf{R}}_{\mathbf{H}}^{k+1}]\!]_T|^2\big)ds\\
&+\tau\sum_{e\in\mathcal{E}_{\mathcal{D}}}\int_e(|\bar{\mathbf{R}}_{\mathbf{H}}^{k+1,int}|+|\mathbf{n}\times\bar{\mathbf{R}}_{\mathbf{E}}^{k+1,int}|)ds\\
\leq&C\tau h^{2p+1}\|\bar{\mathbf{u}}^{k+1}\|^2_{\mathbf{H}^{p+1}(\Omega)}\\
\leq&Ch^{2p+1}\Big(\int_{t_k}^{t_{k+1}}\|\mathbf{u}\|_0^2ds+\frac{\tau^5}{4}\int_{t_k}^{t_{k+1}}\|\mathbf{u}_{tt}\|_0^2ds\Big).
\end{aligned}$$

By Lemma 4.6,

$$\begin{aligned}
\mathcal{A}_8 =&2\int_{t_k}^{t_{k+1}}(\mathcal{J}\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})dt-\tau(\mathcal{J}^{k+1}\mathbf{E}+\mathcal{J}^k\mathbf{E},\bar{\theta}_{\mathbf{E}}^{k+1})\\
&+2\int_{t_k}^{t_{k+1}}(\mathcal{K}\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})dt-\tau(\mathcal{K}^{k+1}\mathbf{H}+\mathcal{K}^k\mathbf{H},\bar{\theta}_{\mathbf{H}}^{k+1})\\
\leq&\frac{\tau\epsilon_0}{10}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^{k}\|_0^2)+\frac{\tau\mu_0}{10}(\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\|\theta_{\mathbf{H}}^{k}\|_0^2)+C\tau^4\int_{t_k}^{t_{k+1}}(\epsilon_0\|\mathbf{E}_t\|_0^2+\mu_0\|\mathbf{H}_t\|_0^2)dt\\
=&\frac{\tau}{10}(\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2)+C\tau^4\int_{t_k}^{t_{k+1}}\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2dt.
\end{aligned}$$

Then we turn to estimate $\mathcal{A}_9$. Actually

$$\begin{aligned}
\mathcal{A}_9 =&\Big(\tau(\mathbf{F}_h^{k+1}+\mathbf{F}_h^k)-2\int_{t_k}^{t_{k+1}}\mathbf{F}dt,\bar{\theta}_{\mathbf{u}}^{k+1}\Big)\\
=&\Big(\int_{t_k}^{t_{k+1}}e^{-\Gamma_e t}dt\mathbf{J}_0-\frac{\tau}{2}(e^{-\Gamma_e t_k}+e^{-\Gamma_e t_{k+1}})\mathbf{J}_0,\theta_{\mathbf{E}}^{k+1}+\theta_{\mathbf{E}}^{k}\Big)\\
&+\Big(\int_{tk}^{t_{k+1}}e^{-\Gamma_m t}dt\mathbf{K}_0-\frac{\tau}{2}(e^{-\Gamma_m t_k}+e^{-\Gamma_m t_{k+1}})\mathbf{K}_0,\theta_{\mathbf{H}}^{k+1}+\theta_{\mathbf{H}}^{k}\Big)\\
\leq&C\tau^5(\|\mathbf{J}_0\|_0^2+\|\mathbf{K}_0\|_0^2)+\frac{\tau\epsilon_0}{10}(\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\|\theta_{\mathbf{E}}^{k}\|_0^2)+\frac{\tau\mu_0}{10}(\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\|\theta_{\mathbf{H}}^{k}\|_0^2)\\
=&C\tau^5(\|\mathbf{J}_0\|_0^2+\|\mathbf{K}_0\|_0^2)+\frac{\tau}{10}(\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2).
\end{aligned}$$

Substituting all those estimates of $\mathcal{A}_j$, $j=1,\cdots,9$, into (4.12), and assuming $\tau\leq 1$, we have

$$\begin{aligned}
&\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2-\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\frac{\tau^2}{4}\epsilon_0\omega_{pe}^2e^{-\Gamma_e\tau}\|\theta_{\mathbf{E}}^{k}\|_0^2\\
&+\frac{\tau^2}{4}\mu_0\omega_{pm}^2\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\frac{\tau^2}{4}\mu_0\omega_{pe}^2e^{-\Gamma_m\tau}\|\theta_{\mathbf{H}}^{k}\|_0^2
\end{aligned}$$

$$
\begin{aligned}
&\le C(T)\tau^2\sum_{j=0}^{k}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{j}\|_0^2+C\tau^6\int_0^{t_{k+1}}(\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_{tt}\|_0^2)ds\\
&\quad+C\tau^4\int_{t_k}^{t_{k+1}}\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|^2ds+\frac{3\tau}{5}(\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k+1}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2)+C\tau^5(\|\mathbf{J}_0\|^2+\|\mathbf{K}_0\|^2)\\
&\quad+C\tau^4\int_{t_k}^{t_{k+1}}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)ds+Ch^{2p+1}\int_{t_k}^{t_{k+1}}\|\mathbf{u}\|_0^2dt\\
&\quad+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\|\theta_{\mathbf{E}}^{k+1}\|_0^2+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\|\theta_{\mathbf{E}}^{k}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\|\theta_{\mathbf{H}}^{k+1}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\|\theta_{\mathbf{H}}^{k}\|_0^2.
\end{aligned}
$$

Summing up the above inequality from $k=0$ to $k=n-1$, we obtain

$$
\begin{aligned}
&\|\boldsymbol{\theta}_{\mathbf{u}}^{n}\|_0^2-\|\boldsymbol{\theta}_{\mathbf{u}}^{0}\|_0^2+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\sum_{k=1}^{n}\|\theta_{\mathbf{E}}^{k}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\sum_{k=1}^{n}\|\theta_{\mathbf{H}}^{k}\|_0^2\\
\le& C(T)\tau\sum_{k=0}^{n-1}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2+C(T)\tau^5\int_0^{t_n}\big(\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_{tt}\|_0^2\big)ds\\
&+C\tau^4\int_0^{t_n}\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|^2ds+\frac{6\tau}{5}\sum_{k=0}^{n-1}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2+C(T)\tau^4(\|\mathbf{J}_0\|^2+\|\mathbf{K}_0\|^2)\\
&+C\tau^4\int_0^{t_n}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)ds+Ch^{2p+1}\int_0^{t_n}\|\mathbf{u}\|_0^2dt+\frac{3}{5}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{n}\|_0^2\\
&+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\sum_{k=1}^{n}\|\theta_{\mathbf{E}}^{k}\|_0^2+\frac{\tau^2\epsilon_0\omega_{pe}^2}{4}\sum_{k=0}^{n-1}\|\theta_{\mathbf{E}}^{k}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\sum_{k=1}^{n}\|\theta_{\mathbf{H}}^{k}\|_0^2+\frac{\tau^2\mu_0\omega_{pm}^2}{4}\sum_{k=0}^{n-1}\|\theta_{\mathbf{H}}^{k}\|_0^2.
\end{aligned}
$$

Considering the fact that $\theta_{\mathbf{u}}^0=0$ and the assumption $\tau\le 1$,

$$
\begin{aligned}
&\frac{2}{5}\|\mathbf{Q}^{\frac{1}{2}}\boldsymbol{\theta}_{\mathbf{u}}^{n}\|_0^2\\
\le& C(T)\tau\sum_{k=0}^{n-1}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{u}}^{k}\|_0^2+C(T)\tau^4(\|\mathbf{J}_0\|_0^2+\|\mathbf{K}_0\|_0^2)+C\tau^4\int_0^{t_n}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)dt\\
&+C(T)\tau^4\int_0^{t_n}(\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_{tt}\|_0^2)ds+Ch^{2p+1}\int_0^{t_n}\|\mathbf{u}\|_0^2dt.
\end{aligned}
$$

By Lemma 4.2,

$$
\begin{aligned}
\|\mathbf{Q}^{\frac{1}{2}}\boldsymbol{\theta}_{\mathbf{u}}^{n}\|_0^2\le& C(T)\tau^4(\|\mathbf{J}_0\|_0^2+\|\mathbf{K}_0\|_0^2)+C(T)\tau^4\int_0^{t_n}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)dt\\
&+C(T)\tau^4\int_0^{t_n}(\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_{tt}\|_0^2)dt+Ch^{2p+1}\int_0^{t_n}\|\mathbf{u}\|_0^2dt.
\end{aligned}
$$

On the other hand,

$$
\|\mathbf{Q}^{\frac{1}{2}}\mathbf{R}_{\mathbf{u}}^{n}\|_0^2\le Ch^{2p+2}\|\mathbf{u}\|_0^2.
$$

By the triangular inequality, we have

$$\begin{aligned}\|\mathbf{Q}^{\frac{1}{2}}e_{\mathbf{u}}^n\|_0^2 \leq & C\tau^4(\|\mathbf{J}_0\|^2+\|\mathbf{K}_0\|^2)+C\tau^4\int_0^{t_n}(\|\nabla\times\mathbf{E}_{tt}\|_0^2+\|\nabla\times\mathbf{H}_{tt}\|_0^2)ds \\ & +C\tau^4\int_0^{t_n}(\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_t\|_0^2+\|\mathbf{Q}^{\frac{1}{2}}\mathbf{u}_{tt}\|_0^2)dt+Ch^{2p+1}\int_0^{t_n}\|\mathbf{u}\|_0^2 dt.\end{aligned}$$

This completes the proof of Theorem 4.2. □

5 Numerical results

In this section, some numerical examples are given to justify our theoretical prediction. According to the theoretical analysis above, we know that our numerical scheme is stable without any restriction on the time step size τ. Actually we obtain accurate numerical solutions when τ is equal to h. In our numerical example, we use rectangular element in 2D and hexahedral element in 3D for simplicity. Moreover, in this section, the L^2-errors are computed in the following way:

$$\|\mathbf{u}-\mathbf{u}_h\|_0=\Big(\sum_{K\in\mathcal{T}_h}\int_K|\mathbf{u}-\mathbf{u}_h|^2 d\mathbf{x}\Big)^{\frac{1}{2}}.$$

5.1 2D numerical example

The similar error estimate for 2-D Maxwell's equations in metamaterials can be obtained in the same way as we have done for 3-D case, by introducing the scalar and vector curl operators

$$curl\mathbf{E}=\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}, \quad \nabla\times E=\Big(\frac{\partial E}{\partial y},-\frac{\partial E}{\partial x}\Big)^T.$$

For simplicity, it is omitted in this paper. We consider the following 2-D model problem

$$\begin{aligned}&\epsilon_0\frac{\partial E_z}{\partial t}-\Big(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\Big)+\mathcal{J}E_z=R_1,\\ &\mu_0\frac{\partial H_x}{\partial t}+\frac{\partial E_z}{\partial y}+K_1=R_2,\\ &\mu_0\frac{\partial H_y}{\partial t}-\frac{\partial E_z}{\partial x}+K_2=R_3,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{J}E_Z=\epsilon_0\omega_{pe}^2\int_0^t e^{-\Gamma_e(t-s)}E_z(\mathbf{x},s)ds,\\ &\mathcal{K}\mathbf{H}=\epsilon_0\omega_{pm}^2\int_0^t e^{-\Gamma_m(t-s)}\mathbf{H}(\mathbf{x},s)ds.\end{aligned}$$

Table 1: The convergence rate of L^2 error for $p=1$, $h=\tau$ (2-D case).

time	time step	h	$\|\mathbf{E}-\mathbf{E}_h\|_0$	order	$\|\mathbf{H}-\mathbf{H}_h\|_0$	order
$T=0.25$	$N=2$	$\frac{1}{8}$	2.7137e-3		3.3446e-3	
	$N=4$	$\frac{1}{16}$	6.4283e-4	2.0777	8.6502e-4	1.9510
	$N=8$	$\frac{1}{32}$	1.5597e-4	2.0431	2.1939e-4	1.9792
	$N=16$	$\frac{1}{64}$	3.8391e-5	2.0224	5.5198e-5	1.9908
$T=0.5$	$N=4$	$\frac{1}{8}$	3.0951e-3		5.8521e-3	
	$N=8$	$\frac{1}{16}$	7.3467e-4	2.0748	1.5065e-3	1.9577
	$N=16$	$\frac{1}{32}$	1.8048e-4	2.0252	3.8290e-4	1.9761
	$N=32$	$\frac{1}{64}$	4.4821e-5	2.0096	9.6589e-5	1.9870
$T=1$	$N=8$	$\frac{1}{8}$	4.3464e-3		5.9869e-3	
	$N=16$	$\frac{1}{16}$	1.0915e-3	1.9935	1.8201e-3	1.7177
	$N=32$	$\frac{1}{32}$	2.7322e-4	1.9982	5.1017e-4	1.8350
	$N=64$	$\frac{1}{64}$	6.8408e-5	1.9978	1.3535e-4	1.9143

Table 2: The convergence rate of L^2 error for $p=2$, $h^2=\tau$ (2-D case).

time	time step	h	$\|\mathbf{E}-\mathbf{E}_h\|_0$	order	$\|\mathbf{H}-\mathbf{H}_h\|_0$	order
$T=0.25$	$N=16$	$\frac{1}{8}$	1.3049e-4		1.7736e-4	
	$N=64$	$\frac{1}{16}$	1.6118e-5	3.0171	2.2552e-5	2.9753
	$N=256$	$\frac{1}{32}$	2.0070e-6	3.0055	2.8569e-6	2.9807
	$N=1024$	$\frac{1}{64}$	2.5049e-7	3.0022	3.5988e-7	2.9889
$T=0.5$	$N=32$	$\frac{1}{8}$	2.0149e-4		2.8271e-4	
	$N=128$	$\frac{1}{16}$	2.5119e-5	3.0039	3.5531e-5	2.9921
	$N=512$	$\frac{1}{32}$	3.1303e-6	3.0044	4.4753e-6	2.9890
	$N=2048$	$\frac{1}{64}$	3.9052e-7	3.0028	5.6214e-7	2.9929
$T=1$	$N=64$	$\frac{1}{8}$	2.4864e-4		3.3709e-4	
	$N=256$	$\frac{1}{16}$	3.0643e-5	3.0204	4.3033e-5	2.9696
	$N=1024$	$\frac{1}{32}$	3.8046e-6	3.0097	5.4331e-6	2.9855
	$N=4096$	$\frac{1}{64}$	4.7407e-7	3.0045	6.8244e-7	2.9930

In our numerical test, we take $\Omega=[0,1]\times[0,1]$, $\epsilon_0=\mu_0=\omega_{pe}=\omega_{pm}=\Gamma_e=\Gamma_m=1$ and R_1, R_2, R_3 such that the exact solution for the 2D version is

$$\begin{pmatrix} E_z \\ H_x \\ H_y \end{pmatrix} = \begin{pmatrix} \sin(\pi x)\sin(\pi y)te^{-t} \\ \sin(\pi x)\cos(\pi y)te^{-t} \\ -\cos(\pi x)\sin(\pi y)te^{-t} \end{pmatrix}.$$

We choose the time step size τ equal to the spatial mesh size h when $p=1$ and τ equal to h^2 when $p=2$. The L^2-errors and their corresponding convergence order are shown in Table 1 for $p=1$ and Table 2 for $p=2$, respectively. It is observed that the convergence rate of both $\mathbf{E}_h$ and $\mathbf{H}_h$ in L^2-norm is $\mathcal{O}(h^{p+1})$, which is better than our theoretical prediction.

5.2 3D numerical example

We consider the following problem in 3D,

$$\epsilon_0\frac{\partial \mathbf{E}}{\partial t}-\nabla\times\mathbf{H}+\mathcal{J}\mathbf{E}=\mathbf{R}_1,\quad \mu_0\frac{\partial \mathbf{H}}{\partial t}+\nabla\times\mathbf{E}+\mathcal{K}\mathbf{H}=\mathbf{R}_2,$$

where

$$\mathcal{J}\mathbf{E}=\epsilon_0\omega_{pe}^2\int_0^t e^{-\Gamma_e(t-s)}\mathbf{E}(\mathbf{x},s)ds,\quad \mathcal{K}\mathbf{H}=\mu_0\omega_{pm}^2\int_0^t e^{-\Gamma_m(t-s)}\mathbf{H}(\mathbf{x},s)ds.$$

In our numerical test, we take $\Omega=[0,1]^3$, $\epsilon_0=\mu_0=\omega_{pe}=\omega_{pm}=\Gamma_e=\Gamma_m=1$ and $\mathbf{R}_1$, $\mathbf{R}_2$ such that the exact solution is

$$\mathbf{E}=te^{-(t+x+y+z)}\begin{pmatrix}(y-y^2)(z-z^2)\\(x-x^2)(z-z^2)\\(x-x^2)(y-y^2)\end{pmatrix},\quad \mathbf{H}=te^{-(t+x+y+z)}\begin{pmatrix}y-z\\z-x\\x-y\end{pmatrix}.$$

We again choose the time step size τ equal to the spatial mesh size h when $p=1$ and τ equal to h^2 when $p=2$. The L^2-errors and their corresponding convergence order are shown in Table 3 for $p=1$ and Table 4 for $p=2$, respectively. It is observed that the convergence rate of both $\mathbf{E}_h$ and $\mathbf{H}_h$ in L^2-norm is $\mathcal{O}(h^{p+1})$, which is also better than our theoretical prediction.

Table 3: The convergence rate of L^2 error for $p=1$, $h=\tau$ (3-D case).

time	time step	h	$\\|\mathbf{E}-\mathbf{E}_h\\|_0$	order	$\\|\mathbf{H}-\mathbf{H}_h\\|_0$	order
$T=0.25$	$N=1$	$\frac{1}{4}$	2.1862e-3		1.5219e-3	
	$N=2$	$\frac{1}{8}$	5.7030e-4	1.9386	4.1673e-4	1.8687
	$N=4$	$\frac{1}{16}$	1.4476e-4	1.9781	1.1042e-4	1.9161
$T=0.5$	$N=2$	$\frac{1}{4}$	2.2365e-3		2.8268e-3	
	$N=4$	$\frac{1}{8}$	5.6052e-4	1.9964	7.7209e-4	1.8723
	$N=8$	$\frac{1}{16}$	1.4050e-4	1.9962	2.0236e-4	1.9318
$T=1$	$N=4$	$\frac{1}{4}$	2.6758e-3		2.4738e-3	
	$N=8$	$\frac{1}{8}$	7.1551e-4	1.9029	6.4479e-4	1.9398
	$N=16$	$\frac{1}{16}$	1.8190e-4	1.9758	1.8337e-4	1.8141

6 Conclusions

In this paper, we developed an implicit discontinuous Galerkin method for time-dependent Maxwell's equations when metamaterials are involved. The spatial discretization is based on DG approach using an upwinding flux. The temporal discretization is an Crank-Nicolson scheme in which the compound trapezoid formula is used for the integral term. Both the unconditionally stability and the convergence rate of $\mathcal{O}(\tau^2+h^{p+1/2})$ were proved. The practical application of this model will be investigated in our future work.

Table 4: The convergence rate of L^2 error for $p=2$, $h^2=\tau$ (3-D case).

time	time step	h	$\|\mathbf{E}-\mathbf{E}_h\|_0$	order	$\|\mathbf{H}-\mathbf{H}_h\|_0$	order
$T=0.25$	$N=1$	$\frac{1}{2}$	4.7354e-4		6.6092e-4	
	$N=4$	$\frac{1}{4}$	7.0842e-5	2.7408	7.4001e-5	3.1588
	$N=16$	$\frac{1}{8}$	8.8542e-6	3.0001	9.1985e-6	2.9807
$T=0.5$	$N=2$	$\frac{1}{2}$	1.0554e-3		9.1744e-4	
	$N=8$	$\frac{1}{4}$	1.1958e-4	3.1417	1.1388e-4	3.0101
	$N=32$	$\frac{1}{8}$	1.4480e-5	3.0458	1.4534e-5	2.9700
$T=1$	$N=4$	$\frac{1}{2}$	1.0823e-3		1.2340e-3	
	$N=16$	$\frac{1}{4}$	1.4790e-4	2.8714	1.4696e-4	3.0698
	$N=64$	$\frac{1}{8}$	1.8546e-5	2.9954	1.8528e-5	2.9876

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