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Strong Convergence Analysis of Split-Step θ-Scheme for Nonlinear Stochastic Differential Equations with Jumps

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Abstract. In this paper, we investigate the mean-square convergence of the split-step θ-scheme for nonlinear stochastic differential equations with jumps. Under some standard assumptions, we rigorously prove that the strong rate of convergence of the split-step θ-scheme in strong sense is one half. Some numerical experiments are carried out to assert our theoretical result.

AMS subject classifications: 65C20, 60H35, 60H10

Key words: Split-step scheme, strong convergence, stochastic differential equation, jump-diffusion, one-side Lipschitz condition.

1 Introduction

We consider *jump-diffusion Itô stochastic differential equations* (JSDEs) of the form

$$\begin{cases} dX(t)=f(X(t_-))dt+g(X(t_-))dW(t)+h(X(t_-))dN(t), \quad t\in(0,T], \\ X(0_-)=X_0, \end{cases} \tag{1.1}$$

where $X(t_-):=\lim_{s\to t-}X(s)$, $f:\mathbb{R}^m\to\mathbb{R}^m$, $g:\mathbb{R}^m\to\mathbb{R}^{m\times d}$ and $h:\mathbb{R}^m\to\mathbb{R}^m$, $m,d\in\mathbb{N}^+$. Here $W(t)$ is a standard d-dimensional Brownian motion, and $N(t)$ is a scalar Poisson process (independent of $W(t)$) with intensity $\lambda>0$, both defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Extension of our work to vector-valued jumps with independent entries is straightforward.

Stochastic differential equations (SDEs) have been widely used in many areas such as chemistry, physics, engineering, biology and mathematical finance to provide models of

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dynamical systems affected by uncertainty factors. When it is the case that a stochastic system is also influenced by some randomly occurring impulses it is often desirable to use a jump-diffusion stochastic model such as (1.1) to characterize these burst phenomena. For more practical applications, one can refer to [1,4,5,7,24].

Since dynamical systems modeled by SDEs rarely admit known explicit solutions, seeking accurate numerical solutions has become a rapidly growing research area. In recent years, much progress has been made in developing numerical methods for solving SDEs [6,8,9,13–15,17,21,22,25]. However, compared with the development of numerical methods for SDEs, numerical methods for solving JSDEs are far from undeveloped, and thus effective and efficient numerical methods are urgently needed. In addition, most of the existing numerical methods for (1.1) are based on globally Lipschitz conditions and linear growth conditions (see, e.g., [1,2,11,18,19,23]) on the coefficients f, g and h. However, these conditions may be too restrictive, which may exclude lots of useful models to be considered, such as some nonlinear problems with super-linearly growing condition coefficients. To relax the conditions, a popular choice is to use one-sided Lipschitz condition on the drift coefficient and globally Lipschitz conditions on the diffusion and jump coefficients [10,12]. Motivated by the above discussions, we aim to design solvers for (1.1) with weaker conditions on the coefficients f, g and h. More precisely, we will theoretically prove that the split-step θ-scheme (see Section 3), admits a one half rate of strong convergence, under the conditions that the drift coefficient f satisfies one-sided Lipschitz condition and the diffusion coefficient g and jump coefficient h satisfy the globally Lipschitz condition.

The rest of this paper is organized as follows. In Section 2, we introduce notations and assumptions. The split-step θ-scheme is introduced in Section 3. In Section 4, we rigorously obtain the boundedness of the solutions of (1.1) and (3.1a). The boundedness will play a key role in our proof of the convergence error estimates of the split-step θ-scheme. Strong convergence estimates are established in Section 5. In Section 6 we present numerical results to validate our theoretical findings. Finally some conclusions are given in Section 7.

2 Notations and assumptions

Throughout the paper, $\langle\cdot,\cdot\rangle$ denotes the scalar inner product in $\mathbb{R}^m$ or $\mathbb{R}^{m\times d}$, and $|\cdot|$ is the associated Euclidean vector norm or Frobenius matrix norm.

We assume the drift coefficient f satisfies the local Lipschitz condition, i.e., for each $R>0$,

$$|f(x)-f(y)|^2\le L_R|x-y|^2 \tag{2.1}$$

for all $x,y\in\mathbb{R}^m$ with $|x|\vee|y|\le R$, and the one-side Lipschitz condition

$$\langle x-y,f(x)-f(y)\rangle\le K_1|x-y|^2 \quad \text{for all } \ x,y\in\mathbb{R}^m, \tag{2.2}$$

while the diffusion and jump coefficients satisfy the global Lipschitz conditions

$$|g(x)-g(y)|^2 \le K_2|x-y|^2 \qquad \text{for all } x,y\in\mathbb{R}^m, \tag{2.3a}$$

$$|h(x)-h(y)|^2 \le K_3|x-y|^2 \qquad \text{for all } x,y\in\mathbb{R}^m. \tag{2.3b}$$

Letting $y=0$ in (2.2), (2.3a), and (2.3b), we can easily derive the following useful estimate (2.4), i.e., there exists a positive constant L such that

$$|\langle x,f(x)\rangle|\vee|g(x)|^2\vee|h(x)|^2 \le L(1+|x|^2) \quad \text{for all } x\in\mathbb{R}^m, \tag{2.4}$$

where

$$L=\max\Big\{\Big(K_1+\frac{1}{2}\Big),\frac{1}{2}|f(0)|^2,2K_2,2|g(0)|^2,2K_3,2|h(0)|^2\Big\}.$$

In order to derive the rate of convergence, we further assume that f grows polynomially. More precisely, we assume that there exist constants $D\in\mathbb{R}^+$ and $q\in\mathbb{Z}^+$ such that for all $x,y\in\mathbb{R}^m$,

$$|f(x)-f(y)|^2 \le D(1+|x|^q+|y|^q)|x-y|^2. \tag{2.5}$$

Note that the condition (2.5) implies the local Lipschitz condition (2.1).

In addition to all of the above assumptions, we also require that the initial data have bounded moments, that is, for any $p>0$ there is a positive constant K_4 such that

$$\mathbb{E}[|X_0|^p] < K_4. \tag{2.6}$$

Before closing this section, we recall that the compensated Poisson process of the Poisson process $N(t)$ is

$$\tilde{N}(t) := N(t)-\lambda t. \tag{2.7}$$

It is a martingale and enjoys the following properties

$$\mathbb{E}[\tilde{N}(t+s)-\tilde{N}(t)]=0, \quad \mathbb{E}[|\tilde{N}(t+s)-\tilde{N}(t)|^2]=\lambda s, \quad t,s\ge 0.$$

The compensated Poisson process will play an important role in our analysis.

3 The split-step θ-scheme

For simplicity, we consider the uniform partition of the interval $[0,T]$ for a given positive integer K. Let $t_n=n\Delta t$ for $n=0,1,\cdots,K$, where $\Delta t=T/K$. Then the split-step θ-scheme for JSDE (1.1) is: given the initial value $Y_0=X_0$, compute $\{Y_n\}_{n=1}^K$ by

$$Y_n^* = Y_n+\Delta t\theta f(Y_n^*), \tag{3.1a}$$

$$Y_{n+1} = Y_n+\Delta t f(Y_n^*)+g(Y_n^*)\Delta W_n+h(Y_n^*)\Delta N_n, \quad n=0,1,\cdots,M-1, \tag{3.1b}$$

for $n=0,1,\cdots,K-1$, where $\theta\in[0,1]$ is a fixed parameter, Y_n is the approximation of $X(t_n)$ at time t_n, $\Delta W_n := W(t_{n+1})-W(t_n)$ and $\Delta N_n =: N(t_{n+1})-N(t_n)$ are the increments of

Brownian motion W_t and the Poisson process N_t, respectively. Notice that, by (3.1a), for each n $(n=0,1,\cdots,K-1)$, Y_n^*, ΔW_n and ΔN_n are independent, and Y_n^* is $\mathcal{F}_{t_n}$ measurable.

In particular, in the case $\theta=0$, the scheme (3.1a)-(3.1b) reduces to the standard Euler scheme for JSDEs, while, if $\theta=1$, the proposed scheme (3.1a)-(3.1b) is equivalent to the split-step backward Euler scheme, which has been introduced and discussed in [10, 12]. From (3.1a), we see that it is an implicit equation when $\theta\in\big(0,1\big]$. We show in the following lemma that (3.1a) admits an unique solution.

Lemma 3.1. *Under the assumptions* (2.2)*, if $K_1\theta\Delta t\le c<1$ holds for some positive constant c, where L is defined in* (2.4)*, the implicit equation* (3.1a) *admits a unique solution with probability one.*

The proof is similar to the proof of Lemma 3.4 in [9], so we omit it here.

For the convenience of discussion, we define the continuous extension $\overline{Y}(t)$ of Y_n on $[t_n,t_{n+1})$ by

$$\overline{Y}(t):=Y_n+(t-t_n)f(Y_n^*)+g(Y_n^*)\Delta W_n(t)+h(Y_n^*)\Delta N_n(t),\quad t\in[t_n,t_{n+1}), \tag{3.2}$$

where $\Delta W_n(t):=W(t)-W(t_n)$ and $\Delta N_n(t):=N(t)-N(t_n)$. Equivalently, we can rewrite (3.2) in the integral form

$$\overline{Y}(t):=Y_0+\int_0^t f(Y^*(s_-))ds+\int_0^t g(Y^*(s_-))dW(s)+\int_0^t h(Y^*(s_-))dN(s), \tag{3.3}$$

where

$$\begin{aligned}Y(s)&:=\sum_{n=0}^{K-1}Y_n1_{\{t_n\le s<t_{n+1}\}}(s)+Y_K1_{\{s=T\}}(s),\\ Y^*(s)&:=\sum_{n=0}^{K-1}Y_n^*1_{\{t_n\le s<t_{n+1}\}}(s)+Y_K^*1_{\{t=T\}}(s),\end{aligned}$$

and 1_F is the characteristic function of a set F, namely,

$$1_F(t)=\begin{cases}0, & t\notin F,\\ 1, & t\in F.\end{cases}$$

Note that $\overline{Y}(t_n)=Y(t_n)=Y_n$, meaning that $\overline{Y}(t)$ and $Y(t)$ coincide with the discrete solutions at the grid-points, hence we can study the error in $\overline{Y}(t)$ in the supremum norm. This will of course give an immediate bound for the error in the discrete approximation.

4 Moment bounds of the exact and numerical solutions

We first introduce some known results for existence, uniqueness, and the moment bound of the exact solution of (1.1), which can be found in [10].

Lemma 4.1. *Under the assumptions* (2.1), (2.2), (2.3a), (2.3b) *and* (2.6), *the JSDE* (1.1) *admits a unique solution X_t on any bounded interval $[0,T]$. Moreover, for each $p>2$, there exists a constant $B_1=B_1(p,T)$ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X(t)|^p\Big]\le B_1(1+\mathbb{E}[|X_0|^p])<\infty. \tag{4.1}$$

Now we turn to show the boundedness of numerical solutions Y_n and Y_n^* of the scheme (3.1a)-(3.1b). Throughout the following analysis, $C=C(p,D,\theta,\lambda,L,T)$ (independent of Δt) denotes a generic constant that may change between occurrences.

Lemma 4.2. *Suppose* (2.2), (2.3a), (2.3b) *and* (2.6) *hold, and let the step size satisfy $\Delta t<\Delta t_0<1/2L$, then for each $p\ge 2$, $1/2\le\theta\le 1$, there exists a constant $B_2=B_2(p,\theta,\lambda,L,T)>0$ (independent of Δt) such that*

$$\mathbb{E}\Big[\sup_{0\le n\Delta t\le T}|Y_n|^{2p}\Big]\bigvee\mathbb{E}\Big[\sup_{0\le n\Delta t\le T}|Y_n^*|^{2p}\Big]\le B_2<\infty, \tag{4.2}$$

where Y_n^* and Y_n are defined in (3.1a)-(3.1b).

Proof. Let N and M be two positive integers satisfying $N\Delta t\le M\Delta t\le T$. By (2.4) and (3.1a) we have

$$\begin{aligned}|Y_n^*|^2&=\langle Y_n,Y_n^*\rangle+\theta\Delta t\langle f(Y_n^*),Y_n^*\rangle\\&\le\frac{1}{2}|Y_n|^2+\frac{1}{2}|Y_n^*|^2+L\theta\Delta t(1+|Y_n^*|^2)\\&\le C_1|Y_n|^2+C_2,\end{aligned}\tag{4.3}$$

where

$$C_1=\frac{1}{1-2L\theta\Delta t_0},\quad C_2=\frac{2L\Delta t_0\theta}{1-2L\theta\Delta t_0}.$$

By squaring both sides of the second equation in (3.1a), we deduce

$$\begin{aligned}|Y_{n+1}|^2=&|Y_n|^2+\Delta t^2(1-2\theta)|f(Y_n^*)|^2+|g(Y_n^*)\Delta W_n|^2+|h(Y_n^*)\Delta N_n|^2+2\Delta t\langle Y_n^*,f(Y_n^*)\rangle\\&+\frac{2}{\theta}\langle Y_n^*,g(Y_n^*)\Delta W_n\rangle+\Big(2-\frac{2}{\theta}\Big)\langle Y_n,g(Y_n^*)\Delta W_n\rangle+\frac{2}{\theta}\langle Y_n^*,h(Y_n^*)\Delta N_n\rangle\\&+\Big(2-\frac{2}{\theta}\Big)\langle Y_n,h(Y_n^*)\Delta N_n\rangle+2\langle g(Y_n^*)\Delta W_n,h(Y_n^*)\Delta N_n\rangle.\end{aligned}$$

Then, for $1/2\le\theta\le 1$, we have by (4.3)

$$\begin{aligned}|Y_{n+1}|^2\le&|Y_n|^2+2\Delta tL(1+|Y_n^*|^2)+|g(Y_n^*)\Delta W_n|^2+|h(Y_n^*)\Delta N_n|^2\\&+\frac{2}{\theta}\langle Y_n^*,g(Y_n^*)\Delta W_n\rangle+\Big(2-\frac{2}{\theta}\Big)\langle Y_n,g(Y_n^*)\Delta W_n\rangle\\&+\frac{2}{\theta}\langle Y_n^*,h(Y_n^*)\Delta N_n\rangle+\Big(2-\frac{2}{\theta}\Big)\langle Y_n,h(Y_n^*)\Delta N_n\rangle\\&+2\langle g(Y_n^*)\Delta W_n,h(Y_n^*)\Delta N_n\rangle\end{aligned}$$

$$
\begin{aligned}
&=|Y_n|^2+A_1\Delta t|Y_n|^2+A_2\Delta t+|g(Y_n^*)\Delta W_n|^2+|h(Y_n^*)\Delta N_n|^2\\
&\quad+A_3\langle Y_n^*,g(Y_n^*)\Delta W_n\rangle+A_4\langle Y_n,g(Y_n^*)\Delta W_n\rangle\\
&\quad+A_3\langle Y_n^*,h(Y_n^*)\Delta N_n\rangle+A_4\langle Y_n,h(Y_n^*)\Delta N_n\rangle\\
&\quad+2\langle g(Y_n^*)\Delta W_n,h(Y_n^*)\Delta N_n\rangle,
\end{aligned}
$$

where $A_1=2LC_1$, $A_2=2L(1+C_2)$, $A_3=2/\theta$, $A_4=(2-2/\theta)$. Adding up the above inequality with respect to n from 0 to $N-1$ leads to

$$
\begin{aligned}
|Y_N|^2\le&|Y_0|^2+A_1\Delta t\sum_{j=0}^{N-1}|Y_j|^2+NA_2\Delta t+\sum_{j=0}^{N-1}|g(Y_j^*)\Delta W_j|^2+\sum_{j=0}^{N-1}|h(Y_j^*)\Delta N_j|^2\\
&+A_3\sum_{j=0}^{N-1}\langle Y_j^*,g(Y_j^*)\Delta W_j\rangle+A_4\sum_{j=0}^{N-1}\langle Y_j,g(Y_j^*)\Delta W_j\rangle+A_3\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta N_j\rangle\\
&+A_4\sum_{j=0}^{N-1}\langle Y_j,h(Y_j^*)\Delta N_j\rangle+2\sum_{j=0}^{N-1}\langle g(Y_j^*)\Delta W_j,h(Y_j^*)\Delta N_j\rangle.
\end{aligned}
$$

Taking the power p on both sides of the above inequality and using the inequality

$$
\Big(\sum_{i=1}^n a_i\Big)^p\le n^{p-1}\sum_{i=1}^n|a_i|^p, \tag{4.4}
$$

we deduce

$$
\begin{aligned}
&\frac{1}{10^{p-1}}|Y_N|^{2p}\\
\le&|Y_0|^{2p}+|A_1\Delta t|^p\Big(\sum_{j=0}^{N-1}|Y_j|^2\Big)^p+|A_2T|^p+\Big(\sum_{j=0}^{N-1}|g(Y_j^*)\Delta W_j|^2\Big)^p\\
&+\Big(\sum_{j=0}^{N-1}|h(Y_j^*)\Delta N_j|^2\Big)^p+|A_3|^p\Big|\sum_{j=0}^{N-1}\langle Y_j^*,g(Y_j^*)\Delta W_j\rangle\Big|^p+|A_4|^p\Big|\sum_{j=0}^{N-1}\langle Y_j,g(Y_j^*)\Delta W_j\rangle\Big|^p\\
&+|A_3|^p\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta N_j\rangle\Big|^p+|A_4|^p\Big|\sum_{j=0}^{N-1}\langle Y_j,h(Y_j^*)\Delta N_j\rangle\Big|^p+2^p\Big|\sum_{j=0}^{N-1}\langle g(Y_j^*)\Delta W_j,h(Y_j^*)\Delta N_j\rangle\Big|^p\\
\le&|Y_0|^{2p}+|A_1|^pT^{p-1}\Delta t\sum_{j=0}^{N-1}|Y_j|^{2p}+|A_2T|^p+\sum_{j=1}^{7}T_{j,N},
\end{aligned}\tag{4.5}
$$

where $T_{j,N}$ is the $(3+j)$th term on the right hand side of the above inequality. Now combining the estimates (A.1)-(A.8) of $T_{j,N}$ $(j=1,\cdots,7)$ (see in Appendix), we obtain

$$
\mathbb{E}\Big[\sup_{0\le N\le M}|Y_N|^{2p}\Big]\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}\Big[\sup_{0\le N\le j}|Y_N|^{2p}\Big].
$$

Then, the boundedness of $\mathbb{E}[\sup_{0\le n\Delta t\le T}|Y_n|^{2p}]$ follows by using the discrete-type Gronwall inequality. The boundedness of $\mathbb{E}[\sup_{0\le n\Delta t\le T}|Y_n^*|^{2p}]$ is obtained thanks to (4.3). $\square$

From the above lemma and the definitions of $Y(t)$ and $Y^*(t)$ in (3.3), we have the following corollary.

Corollary 4.1. *Let $Y(t)$ and $Y^*(t)$ be defined in* (3.3)*. Then*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|Y(t)|^p\Big]\bigvee\mathbb{E}\Big[\sup_{0\le t\le T}|Y^*(t)|^p\Big]\le B_3<\infty,\quad \forall p\ge 2, \tag{4.6}$$

where $B_3=B_2(p,\theta,\lambda,L,T)>0$ is a constant which is independent of Δt.

Similar to the proof of Corollary 3.8 in [9], we can prove

Lemma 4.3. *Suppose the conditions in Lemma* (4.2) *are fulfilled, then we have the estimate*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\overline{Y}(t)|^{2p}\Big]\le B_4<\infty,\quad \forall p\ge 2, \tag{4.7}$$

where $B_4=B_4(p,\theta,\lambda,L,T)$ is a positive constant independent of Δt.

5 Error estimates

Now we are ready to give our convergence result in the following theorem.

Theorem 5.1. *Let $X(t)$ and $\overline{Y}(t)$ be the solution of* (1.1) *and* (3.2)*, respectively. Then under assumptions* (2.1), (2.2), (2.3a), (2.3b), (2.5) *and* (2.6)*, if $1/2\le\theta\le 1$ and $\Delta t<\Delta t_0<1/2L$ with constant L in* (2.4)*, we have the estimate*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\overline{Y}(t)-X(t)|^2\Big]=\mathcal{O}(\Delta t).$$

Proof. Let $e(t):=X(t)-\overline{Y}(t)$. From (1.1) and(3.3), we get

$$\begin{aligned}e(t)=&\int_0^t\big(f(X(s_-))-f(Y^*(s_-))\big)ds+\int_0^t\big(g(X(s_-))-g(Y^*(s_-))\big)dW(s)\\&+\int_0^t\big(h(X(s_-))-h(Y^*(s_-))\big)dN(s).\end{aligned}\tag{5.1}$$

Applying the general Itô formula [4] to $|e(t)|^2$ gives

$$\begin{aligned}|e(t)|^2=&\int_0^t 2\langle f(X(s_-))-f(Y^*(s_-)),e(s_-)\rangle+|g(X(s_-))-g(Y^*(s_-))|^2ds\\&+\int_0^t 2\langle e(s_-),g(X(s_-))-g(Y^*(s_-))dW(s)\rangle\\&+\int_0^t 2\langle e(s_-),h(X(s_-))-h(Y^*(s_-))\rangle+|h(X(s_-))-h(Y^*(s_-))|^2dN(s)\end{aligned}$$

$$
\begin{aligned}
&=\int_0^t 2\langle f(X(s_-))-f(Y^*(s_-)),e(s_-)\rangle+|g(X(s_-))-g(Y^*(s_-))|^2ds\\
&\quad+\lambda\int_0^t 2\langle e(s_-),h(X(s_-))-h(Y^*(s_-))\rangle+|h(X(s_-))-h(Y^*(s_-))|^2ds\\
&\quad+M_1(t)+M_2(t),
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
M_1(t)&=\int_0^t 2\langle e(s_-),g(X(s_-))-g(Y^*(s_-))dW(s)\rangle,\\
M_2(t)&=\int_0^t 2\langle e(s_-),h(X(s_-))-h(Y^*(s_-))\rangle+|h(X(s_-))-h(Y^*(s_-))|^2d\tilde{N}(s).
\end{aligned}
$$

By the assumptions (2.2) and (2.5), we deduce

$$
\begin{aligned}
&\int_0^t \langle f(X(s_-))-f(Y^*(s_-)),e(s_-)\rangle ds\\
=&\int_0^t \langle f(X(s_-))-f(\overline{Y}(s_-)),e(s_-)\rangle ds+\int_0^t \langle f(\overline{Y}(s_-))-f(Y^*(s_-)),e(s_-)\rangle ds\\
\le& C\int_0^t |e(s_-)|^2ds+C\int_0^t |f(\overline{Y}(s_-))-f(Y^*(s_-))|^2+|e(s_-)|^2ds\\
\le& C\int_0^t |e(s_-)|^2ds+C\sup_{0\le s\le t}|\overline{Y}(s)-Y^*(s)|^2\int_0^t(1+|\overline{Y}(s_-)|^q+|Y^*(s_-)|^q)ds.
\end{aligned}
$$

The global Lipschitz condition on g gives us

$$
\begin{aligned}
&\int_0^t |g(X(s_-))-g(Y^*(s_-))|^2ds\\
=&\int_0^t 2|g(X(s_-))-g(\overline{Y}(s_-))|^2ds+\int_0^t 2|g(\overline{Y}(s_-))-g(Y^*(s_-))|^2ds\\
\le& C\int_0^t |e(s_-)|^2ds+C\int_0^t |\overline{Y}(s_-)-Y^*(s_-)|^2ds\\
\le& C\int_0^t |e(s_-)|^2ds+C\sup_{0\le s\le t}|\overline{Y}(s)-Y^*(s)|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_0^t 2\langle f(X(s_-))-f(Y^*(s_-)),e(s_-)\rangle+|g(X(s_-))-g(Y^*(s_-))|^2ds\\
\le& C\int_0^t |e(s_-)|^2ds+C\sup_{0\le s\le t}|\overline{Y}(s)-Y^*(s)|^2\int_0^t(1+|\overline{Y}(s_-)|^q+|Y^*(s_-)|^q)ds\\
&+C\sup_{0\le s\le t}|\overline{Y}(s)-Y^*(s)|^2.
\end{aligned}
\tag{5.3}
$$

Similarly, we have

$$\begin{aligned}
&\lambda\int_0^t 2\langle e(s_-),h(X(s_-))-h(Y^*(s_-))\rangle+|h(X(s_-))-h(Y^*(s_-))|^2ds\\
\le& C\int_0^t |e(s_-)|^2+|h(X(s_-))-h(Y^*(s_-))|^2ds\\
\le& C\int_0^t |e(s_-)|^2ds+C\sup_{0\le s\le t}|\overline{Y}(s)-Y^*(s)|^2.
\end{aligned}\tag{5.4}$$

By the definitions of $\overline{Y}(t)$ and $Y^*(t)$, for $t\in[t_n,t_{n+1})$ we have

$$\begin{aligned}
|\overline{Y}(t)-Y^*(t)|^2=&|(t-t_n-\theta\Delta t)f(Y_n^*)+g(Y_n^*)\Delta W_n(t)+h(Y_n^*)\Delta N_n(t)|^2\\
\le&3\Big((1-\theta)^2\Delta t^2)|f(Y_n^*)|^2+|g(Y_n^*)|^2|\Delta W_n(t)|^2+|h(Y_n^*)|^2|\Delta N_n(t)|^2\Big)\\
\le&C\Big(\Delta t^2(1+|Y_n^*|^q)+(1+|Y_n^*|^2)|\Delta W_n(t)|^2+(1+|Y_n^*|^2)|\Delta N_n(t)|^2\Big).
\end{aligned}$$

Then by Lemma 4.2, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t_n\le t<t_{n+1}}|\overline{Y}(t)-Y^*(t)|^2\Big]\\
=&C\Delta t^2+C\mathbb{E}\Big[\sup_{t_n\le t<t_{n+1}}|\Delta W_n(t)|^2\Big]+C\mathbb{E}\Big[\sup_{t_n\le t<t_{n+1}}|\Delta N_n(t)|^2\Big]\\
\le&C\Delta t.
\end{aligned}\tag{5.5}$$

Combining (5.2), (5.3), (5.4) and (5.5), we deduce

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t}|e(s)|^2\Big]\\
\le&C\int_0^t\mathbb{E}[|e(s_-)|^2]ds+C\Delta t+C\Delta t\int_0^t\mathbb{E}[1+|\overline{Y}(s)|^q+|Y^*(s)|^q]ds+\phi(t)+\psi(t)\\
\le&C\int_0^t\mathbb{E}[|e(s_-)|^2]ds+C\Delta t+\phi(t)+\psi(t),
\end{aligned}\tag{5.6}$$

where

$$\phi(t)=\mathbb{E}\Big[\sup_{0\le s\le t}|M_1(s)|\Big],\quad \psi(t)=\mathbb{E}\Big[\sup_{0\le s\le t}|M_2(s)|\Big].$$

Similar to the proof of Theorem 4.4 in [9], by the Burkhold-Davis-Gundy inequality and the Young inequality [9], we have the following estimates

$$\phi(t)\le\frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le t}|e(s_-)|^2\Big]+C\int_0^t\mathbb{E}[|e(s_-)|^2]ds+C\Delta t,\tag{5.7a}$$

$$\psi(t)\le\frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le t}|e(s_-)|^2\Big]+C\int_0^t\mathbb{E}[|e(s_-)|^2]ds+C\Delta t.\tag{5.7b}$$

Inserting (5.7) into (5.6) yields

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le s\le t}|e(s)|^2\Big] &\le C\int_0^t \mathbb{E}[|e(s_-)|^2]ds+C\Delta t\\ &\le C\int_0^t \mathbb{E}\Big[\sup_{0\le r\le s}|e(r_-)|^2\Big]ds+C\Delta t.\end{aligned}$$

The proof is completed by using Gronwall lemma. □

Remark 5.1. The obtained error estimates hold true for nonuniform time partition.

Remark 5.2. The schemes introduced in [10, 12] is a special case of our split-step backward Euler scheme with $\theta=1$, i.e., our scheme is more general. The proof of convergence error estimate result is different from those used in [10,12], in which the authors got their estimates by changing their scheme to a Euler scheme for a modified JSDE. In this paper, we deduced our error estimates by introducing a extended time-continuous JSDE. Our proof is more direct and much simpler than that used in [10, 12].

6 Numerical experiments

In this section, we will present some numerical examples to show the properties of the split-step θ-scheme.

In our numerical experiments, the errors e between exact solutions and numerical solutions are measured by

$$e:=\hat{\mathbb{E}}[|X(T)-Y_k|]:=\frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}|X^{(i)}(T)-Y_K^{(i)}|,$$

where the positive integer N_{mc} is the sample times in numerical tests, $Y_K^{(i)}$ is the numerical approximation solution at the time $t_k=T$ by our split-step θ-scheme (3.1a)-(3.1b) at the ith sampling. Note that the $\hat{\mathbb{E}}[|X(T)-Y_k|]$ is the Monte-Carlo approximation of the mathematical expectation $\mathbb{E}[|X(T)-Y_k|]$.

We choose the jump intensity $\lambda=1$, the initial value $X_0=1$, and the sampling times number $N_{mc}=5000$ in our numerical simulations.

In order to show the performance of the split-step θ-scheme (3.1a)-(3.1b), we consider two different types of examples respectively. For one type, all the coefficients f, g and h in (1.1) satisfy global Lipschitz conditions. For the other, the coefficient f satisfies the one-side Lipchitz condition, and the coefficients g and h are globally Lipschitz continuous. Further, in each example, we simulate several different cases in order to detect the effects of the parameter θ in the split-step θ-scheme (3.1a)-(3.1b) and the stochastic noises in the models on the proposed scheme.

6.1 Numerical example I

Considering the first type example, for simplicity, we use the linear JSDE

$$\begin{cases} dX(t)=aX(t_-)dt+bX(t_-)dW_t+cX(t_-)dN(t), & t\in(0,T], \\ X(0)=X_0, \end{cases} \tag{6.1}$$

where a, b and c are real numbers. It is well known that the exact solution of (6.1) is

$$X(t)=X(0)e^{(a-\frac{1}{2}b^2)t+bW(t)}(1+c)^{N(t)}.$$

To test the effects of stochastic noises in the model on the split-step θ-scheme (3.1a)-(3.1b), we consider various cases.

We list the errors e and the convergence rates CR with different θ's in the split-step θ-scheme (3.1a)-(3.1b) for five sets of the parameters in Tables 1-5.

Table 1: Errors e and convergence rates CR with $a=1$, $b=1$, $c=1$, $T=1$.

Δt	2^{-5}	2^{-6}	2^{-7}	2^{-8}	CR
$\theta=0$	1.0391	0.7842	0.5398	0.3784	0.5149
$\theta=0.25$	1.0265	0.7109	0.5235	0.3550	0.4924
$\theta=0.5$	1.0901	0.7701	0.5319	0.3882	0.5147
$\theta=0.75$	1.0658	0.7400	0.5124	0.3549	0.4916
$\theta=1$	1.0811	0.7724	0.5418	0.3913	0.5045

Table 2: Errors e and convergence rates CR with $a=1$, $b=0.01$, $c=1$, $T=1$.

Δt	2^{-5}	2^{-6}	2^{-7}	2^{-8}	CR
$\theta=0$	0.4076	0.2053	0.1079	0.0570	0.9710
$\theta=0.25$	0.3719	0.1830	0.0869	0.0367	1.0377
$\theta=0.5$	0.2118	0.1046	0.0460	0.0256	0.8559
$\theta=0.75$	0.1609	0.0568	0.0255	0.0107	1.1064
$\theta=1$	0.2367	0.1211	0.0687	0.0342	1.1344

Table 3: Errors e and convergence rates CR with $a=1$, $b=1$, $c=0.01$, $T=1$.

Δt	2^{-5}	2^{-6}	2^{-7}	2^{-8}	CR
$\theta=0$	0.2758	0.2030	0.1425	0.1007	0.4941
$\theta=0.25$	0.2635	0.1815	0.1305	0.0951	0.4947
$\theta=0.5$	0.2814	0.1940	0.1385	0.0995	0.4972
$\theta=0.75$	0.2820	0.1946	0.1379	0.0990	0.5041
$\theta=1$	0.2937	0.2036	0.1396	0.0997	0.5002

6.2 Numerical example II

For the second type of example, we consider the nonlinear JSDE as follows.

$$dX(t)=\Big(\Big(\eta+\frac{1}{2}\sigma^2\Big)X(t_-)+\kappa X(t_-)^3\Big)dt+\sigma X(t_-)dW_t+\gamma X(t_-)dN(t) \tag{6.2}$$

Table 4: Errors e and convergence rates CR with $a=1$, $b=0.01$, $c=0.01$, $T=1$.

Δt	2^{-5}	2^{-6}	2^{-7}	2^{-8}	CR
$\theta=0$	0.0425	0.0216	0.0109	0.0055	0.9951
$\theta=0.25$	0.0216	0.0109	0.0055	0.0028	0.9961
$\theta=0.5$	0.0005070	0.0002734	0.0001437	0.0000742	0.9184
$\theta=0.75$	0.0217	0.0107	0.0053	0.0027	1.0040
$\theta=1$	0.0442	0.0218	0.0108	0.0054	1.0050

Table 5: Errors e and convergence rates CR with $a=1$, $b=0.0001$, $c=0.0001$, $T=1$.

Δt	2^{-5}	2^{-6}	2^{-7}	2^{-8}	CR
$\theta=0$	0.0413	0.0209	0.0105	0.0053	0.9952
$\theta=0.25$	0.0208	0.0105	0.0053	0.0026	0.9962
$\theta=0.5$	0.0002170	0.0000532	0.0000128	0.0000029	1.7574
$\theta=0.75$	0.0217	0.0107	0.0053	0.0027	1.0038
$\theta=1$	0.0437	0.0215	0.0107	0.0053	1.0050

Table 6: The errors e and the convergence rates CR with $\eta=0.5$, $\kappa=-1$, $\sigma=2$, $\gamma=1$, $T=1$.

Δt	2^{-11}	2^{-12}	2^{-13}	2^{-14}	CR
$\theta=0$	0.0199	0.0137	0.0096	0.0066	0.5266
$\theta=0.25$	0.0199	0.0138	0.0097	0.0066	0.5252
$\theta=0.5$	0.0198	0.0138	0.0096	0.0067	0.5207
$\theta=0.75$	0.0201	0.0140	0.0098	0.0066	0.5299
$\theta=1$	0.0201	0.0139	0.0097	0.0068	0.5233

Table 7: The errors e and the convergence rates CR with $\eta=0.5$, $\kappa=-1$, $\sigma=0.02$, $\gamma=1$, $T=1$.

Δt	2^{-11}	2^{-12}	2^{-13}	2^{-14}	CR
$\theta=0$	0.0002549	0.0001275	0.0000688	0.0000386	0.9055
$\theta=0.25$	0.0001761	0.0000933	0.0000488	0.0000264	0.9150
$\theta=0.5$	0.0002450	0.0001235	0.0000643	0.0000336	0.9543
$\theta=0.75$	0.0004535	0.0002239	0.0001106	0.0000542	1.0215
$\theta=1$	0.0006264	0.0003117	0.0001519	0.0000745	1.0252

for $t\in(0,T]$ with initial condition $X(0)=X_0$, where η, κ, σ and γ are real parameters. In the following experiments, we choose different η, κ, σ and γ such that conditions in Section 2 hold. Since the exact solution of (6.2) can not be expressed in explicit closed form, we take split-step θ-scheme (3.1a)-(3.1b) with a very small time step, we choose $\Delta t=2^{-18}$, as our exact reference solution.

Consider six sets of parameters for the same purpose as Numerical example I, and display the errors e and the convergence rates CR with different θ's in Tables 6-11.

From all the results listed in above Tables 1-11, we conclude that:

1. The convergence rate of the split-step θ-scheme (3.1a)-(3.1b) is 0.5 for solving general JSDEs. This one half convergence rate result is consistent with our theoretical one in Theorem 5.1;

Table 8: The errors e and the convergence rates CR with $\eta=0.5$, $\kappa=-1$, $\sigma=2$, $\gamma=0.01$, $T=1$.

Δt	2^{-11}	2^{-12}	2^{-13}	2^{-14}	CR
$\theta=0$	0.0171	0.0116	0.0082	0.0057	0.5218
$\theta=0.25$	0.0172	0.0120	0.0085	0.0057	0.5260
$\theta=0.5$	0.0172	0.0119	0.0083	0.0058	0.5197
$\theta=0.75$	0.0173	0.0121	0.0085	0.0058	0.5255
$\theta=1$	0.0168	0.0119	0.0084	0.0058	0.5133

Table 9: The errors e and the convergence rates CR with $\eta=0.5$, $\kappa=-1$, $\sigma=0.02$, $\gamma=0.01$, $T=1$.

Δt	2^{-11}	2^{-12}	2^{-13}	2^{-14}	CR
$\theta=0$	0.00003621	0.00001797	0.00000885	0.00000431	1.0233
$\theta=0.25$	0.00001730	0.00000861	0.00000427	0.00000212	1.0095
$\theta=0.5$	0.000004650	0.000002692	0.000001646	0.000001041	0.7185
$\theta=0.75$	0.00002081	0.00001034	0.00000510	0.00000250	1.0190
$\theta=1$	0.00003972	0.00001971	0.00000963	0.00000466	1.0305

Table 10: The errors e and the convergence rates CR with $\eta=0.5$, $\kappa=-1$, $\sigma=0.000002$, $\gamma=0.000001$, $T=1$.

Δt	2^{-11}	2^{-12}	2^{-13}	2^{-14}	CR
$\theta=0$	3.7080E-05	1.8392E-05	9.0494E-06	4.3786E-06	1.0269
$\theta=0.25$	1.8537E-05	9.1952E-06	4.5245E-06	2.1892E-06	1.0269
$\theta=0.5$	6.1899E-09	1.5881E-09	4.1575E-10	1.1329E-10	1.9249
$\theta=0.75$	1.8542E-05	9.1964E-06	4.5249E-06	2.1894E-06	1.0270
$\theta=1$	3.7062E-05	1.8387E-05	9.0483E-06	4.3784E-06	1.0267

Table 11: The errors e and the convergence rates CR with $\eta=0.5$, $\kappa=-1$, $\sigma=0.02$, $\gamma=1$, $T=100$.

Δt	2^{-11}	2^{-12}	2^{-13}	2^{-14}	CR
$\theta=0$	NaN	NaN	0.0107	0.0034	–
$\theta=0.25$	0.0189	0.0088	0.0044	0.0021	1.0502
$\theta=0.5$	0.0347	0.0162	0.0076	0.0035	1.1006
$\theta=0.75$	0.0508	0.0238	0.0118	0.0057	1.0503
$\theta=1$	0.0693	0.0310	0.0160	0.0080	1.0330

2. Table 1 and Table 2, Table 6 and Table 7 show that the accuracy of the scheme will increase and the convergence rate of the scheme will rise to order 1 when the diffusion coefficient g becomes small with the jump coefficient h fixed. And Table 1 and Table 3, Table 6 and Table 8 show that the accuracy and the convergence rate are improved little when the jump coefficient h becomes smaller with the diffusion coefficient g fixed;

3. Taking Table 1 and Table 4, Table 6 and Table 9 into consideration, we find that the accuracies are highly improved and the convergence rates can reach order 1 when the diffusion coefficient g and the jump coefficient h become smaller simultaneously, and furthermore when the coefficients are very small, the convergence rate of the scheme (3.1a)-(3.1b) with $\theta=0.5$ becomes 2, which is the same as the

Crank-Nicolson scheme for solving deterministic ODEs. These observations are interesting since the coefficients are often small in many realistic applications. We will analyze the split-step θ-scheme (3.1a)-(3.1b) for JSDEs with small noises in our future work;

4. The split-step θ-scheme (3.1a)-(3.1b) with $\theta=1$ is equivalent to the split-step backward Euler scheme considered in [10,12], and in this case, our numerical and theoretical results coincide with those obtained in [10,12];
5. From all the results listed in above tables, the $\theta=0.5$ should be a good choice for the split-step scheme (3.1a)-(3.1b) for JSDEs, especially for JSDEs with small noises.

7 Conclusions

In this paper we studied the split-step θ-scheme for nonlinear stochastic differential equations with jumps. Under relatively weaker conditions, we proved that the proposed scheme enjoys strong convergence of order $1/2$. Particularly, in the case $\theta=1$, the proposed scheme becomes split-step backward Euler scheme, which is introduced and discussed in [10, 12]. The theoretical results are conformed by various experiments. The experiment results also show that it is more accurate for solving JSDEs with small noises when $\theta=0.5$ is used in the scheme. In the future work, we will consider the effects of the noise coefficients on the error estimates of the split-step θ-scheme for JSDEs.

Appendix: Estimates of the terms $T_{j,N}$ $(j=1,\cdots,7)$

Estimates of the terms $T_{j,N}$ $(j=1,\cdots,7)$.

(1) The estimate of $T_{1,N}$. By (4.4), we obtain the estimate

$$T_{1,N}\le N^{p-1}\Big(\sum_{j=0}^{N-1}|g(Y_j^*)\Delta W_j|^{2p}\Big).$$

Then combining with (2.4) and (4.3), we deduce

$$\begin{aligned}\mathbb{E}\Big[\sup_{1\le N\le M}T_{1,N}\Big]\le& M^{p-1}\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)\Delta W_j|^{2p}\Big]\\ \le& M^{p-1}\sum_{j=0}^{M-1}\mathbb{E}[|g(Y_j^*)|^{2p}]\mathbb{E}[|\Delta W_j|^{2p}]\\ \le& CM^{p-1}\Delta t^p\sum_{j=0}^{M-1}\mathbb{E}[(1+|Y_j^*|)^{2p}]\\ \le& C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}].\end{aligned}\tag{A.1}$$

(2) The estimate of $T_{2,N}$. By the definition of $T_{2,N}$, we have the following estimate

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{1\le N\le M} T_{2,N}\Big]\\
\le& C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)\Delta\tilde{N}_j|^2\Big)^p\Big]+C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)\Delta t|^2\Big)^p\Big]\\
\le& C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)|^2(\Delta\tilde{N}_j^2-2\tilde{N}_j\Delta\tilde{N}_j)\Big)^p\Big]+C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)\Delta t|^2\Big)^p\Big]\\
\le& C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)|^2\check{N}(t)\Big)^p\Big]+C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)|^2\tilde{N}_j\Delta\tilde{N}_j\Big)^p\Big]\\
&+C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)|^2\Delta t\Big)^p\Big]+C\mathbb{E}\Big[\sup_{0\le N\le M}\Big(\sum_{j=0}^{N-1}|h(Y_j^*)\Delta t|^2\Big)^p\Big],
\end{aligned}$$

where $\tilde{N}(t)=N(t)-\lambda t$ and $\check{N}(t)=\tilde{N}(t)^2-\lambda t$. By the definition of $N(t)$, the processes $\tilde{N}(t)$ and $\check{N}(t)$ are all martingales. Using the Burkhold-Davis-Gundy inequality [16] and the inequality (4.4), we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{1\le N\le M} T_{2,N}\Big]\le& C\mathbb{E}\Big[\sum_{j=0}^{M-1}|h(Y_j^*)|^4\Delta t\Big]^{p/2}+C\mathbb{E}\Big[\sum_{j=0}^{M-1}|h(Y_j^*)|^4\tilde{N}_j^2\Delta t\Big]^{p/2}\\
&+C\Delta t^pM^{p-1}\sum_{j=0}^{M-1}\mathbb{E}[|h(Y_j^*)|^{2p}]+C\Delta t^{2p}M^{p-1}\sum_{j=0}^{M-1}\mathbb{E}[|h(Y_j^*)|^{2p}]\\
\le& C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|h(Y_j^*)|^{2p}]+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|h(Y_j^*)|^{2p}\tilde{N}_j^p].
\end{aligned}$$

Then by the assumption (2.4) and the estimate

$$\begin{aligned}
\mathbb{E}[|h(Y_j^*)|^{2p}\tilde{N}_j^p]=&\sum_{i=0}^{\infty}\mathbb{E}[\mathbb{E}[|h(Y_j^*)|^{2p}(i-\lambda t_j)^p|N_j=i]]P(N_j=i)\\
=&\sum_{i=0}^{\infty}\mathbb{E}[\mathbb{E}[|h(Y_j^*)|^{2p}|N_j=i]]\frac{e^{-\lambda t_j}(\lambda t_j)^i}{i!}(i-\lambda t_j)^p\\
=&\mathbb{E}[|h(Y_j^*)|^{2p}]\sum_{i=0}^{\infty}\frac{e^{-\lambda t_j}(\lambda t_j)^i}{i!}(i-\lambda t_j)^p\le C\mathbb{E}[|h(Y_j^*)|^{2p}]\\
\le& C+C\mathbb{E}[|Y_j|^{2p}],
\end{aligned}$$

we obtain the estimate

$$\mathbb{E}\Big[\sup_{1\le N\le M} T_{2,N}\Big]\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]. \tag{A.2}$$

(3) The estimates of $T_{3,N}$ and $T_{4,N}$. Using together the Burkhold-Davis-Gundy inequality and the inequality (4.4) yields

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le N\le M} T_{3,N}\Big] \le & C\mathbb{E}\Big[\sum_{j=0}^{M-1}|Y_j^*|^2|g(Y_j^*)|^2\Delta t\Big]^{p/2}\\ \le & C\Delta t^{p/2}M^{p/2-1}\mathbb{E}\Big[\sum_{j=0}^{M-1}|Y_j^*|^p(1+|Y_j^*|^2)^{p/2}\Big]\\ \le & C\Delta t^p\sum_{j=0}^{M-1}[1+\mathbb{E}|Y_j^*|^{2p}]\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]. \end{aligned}\tag{A.3}$$

Similarly, we have

$$\mathbb{E}\Big[\sup_{0\le N\le M} T_{4,N}\Big]\le C\mathbb{E}\Big[\sum_{j=0}^{M-1}|Y_j|^2|g(Y_j^*)|^2\Delta t\Big]^{p/2}\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]. \tag{A.4}$$

(4) The estimates of $T_{5,N}$ and $T_{6,N}$. We have the estimate

$$\begin{aligned}\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta N_j\rangle\Big|^p \le & C\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta\tilde{N}_j\rangle\Big|^p+C\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta t\rangle\Big|^p\\ \le & C\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta\tilde{N}_j\rangle\Big|^p+C\Delta t^p\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\rangle\Big|^p.\end{aligned}\tag{A.5}$$

Using together the assumption (2.3b), the Burkhold-Davis-Gundy inequality and the inequality (4.4) gives

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le N\le M}\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta\tilde{N}_j\rangle\Big|^p\Big]\\ \le & C\mathbb{E}\Big[\sum_{j=0}^{M-1}|Y_j^*|^2|h(Y_j^*)|^2\Delta t\Big]^{p/2}\\ \le & C\Delta t^{p/2}M^{p/2-1}\mathbb{E}\Big[\sum_{j=0}^{M-1}|Y_j^*|^p(1+|Y_j^*|^2)^{p/2}\Big]\\ \le & C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}].\end{aligned}$$

Thus we deduce

$$\mathbb{E}\Big[\sup_{0\le N\le M} T_{5,N}\Big]\le C\mathbb{E}\Big[\sup_{0\le N\le M}\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\Delta\tilde{N}_j\rangle\Big|^p\Big]+C\Delta t^pC\mathbb{E}\Big[\sup_{0\le N\le M}\Big|\sum_{j=0}^{N-1}\langle Y_j^*,h(Y_j^*)\rangle\Big|^p\Big]$$

$$
\begin{aligned}
&\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]+C\Delta t^p M^{p-1}\mathbb{E}\Big[\sum_{j=0}^{M-1}|Y_j^*|^p(1+|Y_j^*|^2)^{p/2}\Big]\\
&\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]. \qquad\qquad (A.6)
\end{aligned}
$$

Similarly, we have the estimate

$$
\mathbb{E}\Big[\sup_{0\le N\le M} T_{6,N}\Big]\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]. \tag{A.7}
$$

(5) The estimate of $T_{7,N}$. Finally, by using the independence of Brownian motion $W(t)$ and the Poisson process $N(t)$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le N\le M} T_{7,N}\Big]\\
\le& C\mathbb{E}\Big[\sup_{0\le N\le M}\Big|\sum_{j=0}^{N-1}\langle g(Y_j^*)\Delta W_j, h(Y_j^*)\Delta\tilde{N}_j\rangle\Big|^p\Big]+C\mathbb{E}\Big[\sup_{0\le N\le M}\Big|\sum_{j=0}^{N-1}\langle g(Y_j^*)\Delta W_j, h(Y_j^*)\Delta t\rangle\Big|^p\Big]\\
\le& C\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)|^2|h(Y_j^*)|^2\Delta t\Big]^{p/2}+C\Delta t^p\mathbb{E}\Big[\sup_{0\le N\le M}\Big|\sum_{j=0}^{N-1}\langle g(Y_j^*)\Delta W_j, h(Y_j^*)\rangle\Big|^p\Big]\\
\le& C\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)|^2|h(Y_j^*)|^2\Delta t\Big]^{p/2}+C\Delta t^p\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)|^2|h(Y_j^*)|^2\Delta t\Big]^{p/2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le N\le M} T_{7,N}\Big]\\
\le& C\Delta t^{p/2}M^{p/2-1}\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)|^p|h(Y_j^*)|^p\Big]+C\Delta t^{3p/2}M^{p/2-1}\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)|^p|h(Y_j^*)|^p\Big]\\
\le& C\Delta t\mathbb{E}\Big[\sum_{j=0}^{M-1}|g(Y_j^*)|^p|h(Y_j^*)|^p\Big]\\
\le& C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}[|Y_j|^{2p}]. \qquad\qquad (A.8)
\end{aligned}
$$

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