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New Non-Travelling Wave Solutions of Calogero Equation

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Abstract. In this paper, the idea of a combination of variable separation approach and the extended homoclinic test approach is proposed to seek non-travelling wave solutions of Calogero equation. The equation is reduced to some $(1+1)$-dimensional nonlinear equations by applying the variable separation approach and solves reduced equations with the extended homoclinic test technique. Based on this idea and with the aid of symbolic computation, some new explicit solutions can be obtained.

AMS subject classifications: 35C09, 35Q53, 68W30

Key words: Variable separation approach, extended homoclinic test approach, non-travelling wave solution.

1 Introduction

Nonlinear evolution equations are related to nonlinear phenomena in nonlinear science such as physics, mechanics, biology and chemistry. To further explain some physical phenomena, seeking exact solutions of nonlinear evolution equations is of great significance. It is well known that the method of variable separation is one of the most universal and efficient means for studying linear partial differential equations (PDEs). Several methods of variable separation for nonlinear PDEs have been suggested, such as the ansatz-based method [1], the formal variable separation approach [2], the functional variable separation approach [3, 4], the derivative-dependent functional variable separation approach [5, 6] and the multi-linear variable separation approach [7–9].

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In this paper, we consider the following $(2+1)$-dimensional breaking soliton equation

$$u_{xt}-4u_xu_{xy}-2u_yu_{xx}+u_{xxxy}=0, \tag{1.1}$$

which was first presented by Calogero and Degasperis [10, 11]. This equation was also constructed by Bogoyavlenskii and Schiff in different ways [12–14]. The equation is used to describe the interaction of a Riemann wave propagating along the y axis with a long wave along the x axis. The Plainlevé property, Darboux covariant Lax pairs, infinite conservation laws, Hamilton structure and the Lax pair of Eq. (1.1) have been discussed by many researchers in [15–17]. The bilinear Bäcklund transformation, nonlinear superposition formula and Wronskian determinant solution have been discussed in [18]. Moreover, a considerable number of exact specific solutions have been developed and can be found in [19–25].

The goal of the present work is to investigate the non-travelling wave solutions for Eq. (1.1) by using the multi-linear variable separation approach combining with the extended homoclinic test approach [26]. First, we apply the method of variable separation to reduce Eq. (1.1) to some $(1+1)$-dimensional nonlinear equation. Then, solving the reduced equation by the extended homoclinic test technique with the aid of Maple, we obtain some new non-travelling wave explicit solutions of Eq. (1.1). These solutions can provide an important practical check on the accuracy and reliability of such integrators.

2 The non-travelling wave solutions

In this section, we employ the method of separation of variables together with the extended homoclinic test technique solving the Calogero Equation (1.1). We assume that the solutions for Eq. (1.1) are of the x-line form

$$u(x,y,t)=\varphi(\xi,t)+q(y,t), \tag{2.1}$$

where $\xi=px+\theta(y,t)$.

Substituting (2.1) into (1.1), one obtains

$$\varphi_{\xi t}+(\theta_t-2pq_y)\varphi_{\xi\xi}-6p\theta_y\varphi_\xi\varphi_{\xi\xi}+p^2\theta_y\varphi_{\xi\xi\xi\xi}=0. \tag{2.2}$$

In order to simplify Eq. (2.2), we assume that

$$\theta_t-2pq_y=0. \tag{2.3}$$

Now we employ the method of separation of variables solving Eq. (2.3).

First we seek the multiplicative separable solution

$$\theta=f(t)g(y), \tag{2.4}$$

where $f(t)$, $g(y)$ are smooth functions to be determined later.

Inserting (2.4) into (2.3), we get

$$q(y,t)=\frac{f'(t)\int g(y)dy}{2p}. \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.2), we have

$$\varphi_{\xi t}-6pf(t)g'(y)\varphi_{\xi}\varphi_{\xi\xi}+p^2f(t)g'(y)\varphi_{\xi\xi\xi\xi}=0. \tag{2.6}$$

Let $g'(y)=$constant, then we suppose without loss of generality that $g(y)=y$. Eq. (2.6) reduces to the following variable coefficient equation

$$\varphi_{\xi t}-6pf(t)\varphi_{\xi}\varphi_{\xi\xi}+p^2f(t)\varphi_{\xi\xi\xi\xi}=0. \tag{2.7}$$

We make the transformation

$$\varphi(\xi,t)=v(\xi,\eta), \quad \eta=\int f(t)dt. \tag{2.8}$$

Substituting (2.8) into (2.7) yields a partial differential equation with constant coefficients

$$v_{\xi\eta}-6pv_{\xi}v_{\xi\xi}+p^2v_{\xi\xi\xi\xi}=0. \tag{2.9}$$

Integrating (2.9) once with respect to ξ and discarding the integration constant yield

$$v_{\eta}-3pv_{\xi}^2+p^2v_{\xi\xi\xi}=0. \tag{2.10}$$

To solve Eq. (2.10), we introduce a transformation

$$v=-2p(\ln\phi)_{\xi}. \tag{2.11}$$

Substituting (2.11) into (2.10) yields

$$(D_{\xi}D_{\eta}+p^2D_{\xi}^4)\phi\cdot\phi=0. \tag{2.12}$$

With regard to (2.12), using the extended homoclinic test approach [26], we seek the solution in the form

$$\phi=p_1\cos(\xi_1)+p_2\exp(\xi_2)+\exp(-\xi_2), \tag{2.13}$$

where $\xi_1=a_1\xi+c_1\eta$, $\xi_2=a_2\xi+c_2\eta$, a_i, c_i, $(i=1,2)$ are arbitrary constants to be determined later.

Substituting (2.13) into (2.12) and equating all coefficients of $\sin(\xi_1)$, $\cos(\xi_1)$, $\exp(\xi_2)$, $\exp(-\xi_2)$ to zero, one yields the following set of algebraic equation for a_i, c_i, p_i, $(i=1,2)$

$$\begin{cases} p_1p_2(a_1c_2+a_2c_1+4p^2a_1a_2^3-4p^2a_1^3a_2)=0,\\ p_1(a_1c_2+a_2c_1+4p^2a_1a_2^3-4p^2a_1^3a_2)=0,\\ p_1p_2(a_2c_2-a_1c_1+p^2a_1^4+p^2a_2^4-6p^2a_1^2a_2^2)=0,\\ p_1(a_2c_2-a_1c_1+p^2a_1^4+p^2a_2^4-6p^2a_1^2a_2^2)=0,\\ 4p^2p_1^2a_1^4+4p_2a_2c_2-p_1^2a_1c_1+16p^2p_2a_2^4=0. \end{cases} \tag{2.14}$$

Solving Eq. (2.14) with the aid of Maple, one gets the following results:

Case 1:

$$p_1=0, \quad a_1=a_1, \quad c_1=c_1, \quad p_2=p_2, \quad a_2=a_2, \quad c_2=-4p^2a_2^3. \tag{2.15}$$

From (2.13), (2.15), (2.11) and (2.1), we get

$$u=-2pa_2\frac{p_2\exp(\xi_2)-\exp(-\xi_2)}{p_2\exp(\xi_2)+\exp(-\xi_2)}+\frac{f'(t)y^2}{4p}. \tag{2.16}$$

When $p_2>0$, one obtains a kinky non-traveling wave solution

$$u_1=-2pa_2\tanh\Big(\xi_2+\frac{1}{2}\ln p_2\Big)+\frac{f'(t)y^2}{4p}. \tag{2.17}$$

When $p_2<0$, one obtains a kinky non-traveling wave solution

$$u_2=-2pa_2\coth\Big[\xi_2+\frac{1}{2}\ln(-p_2)\Big]+\frac{f'(t)y^2}{4p}. \tag{2.18}$$

Case 2:

$$\begin{cases} a_1=a_1,\ a_2=a_2,\ p_1=p_1,\ p_2=-\dfrac{p_1^2a_1^2}{4a_2^2}, \\ c_1=a_1(p^2a_1^2-3p^2a_2^2),\ c_2=a_2(3p^2a_1^2-p^2a_2^2). \end{cases} \tag{2.19}$$

Combining (2.13), (2.19), (2.11) with (2.1) yields

$$u=2p\frac{a_1p_1\sin(\xi_1)+a_2(\frac{p_1^2a_1^2}{4a_2^2}\exp(\xi_2)+\exp(-\xi_2))}{p_1\cos(\xi_1)-\frac{p_1^2a_1^2}{4a_2^2}\exp(\xi_2)+\exp(-\xi_2)}+\frac{f'(t)y^2}{4p}, \tag{2.20}$$

where

$$\xi_1=a_1(px+f(t)y)+a_1(p^2a_1^2-3p^2a_2^2)\int f(t)dt,$$
$$\xi_2=a_2(px+f(t)y)+a_2(3p^2a_1^2-p^2a_2^2)\int f(t)dt.$$

Let $p_1a_1a_2>0$, then (2.20) can be written as follows

$$u_3=2a_1a_2p\frac{\sin(\xi_1)+\cosh(\xi_2+\theta_1)}{a_2\cos(\xi)-a_1\sinh(\xi_2+\theta_1)}+\frac{f'(t)y^2}{4p}, \tag{2.21}$$

where

$$\theta_1=\ln\Big(\frac{p_1a_1}{2a_2}\Big).$$

Let $p_1a_1a_2<0$, then (2.20) can be written as follows

$$u_4=2a_1a_2p\frac{\sin(\xi_1)-\cosh(\xi_2+\theta_2)}{a_2\cos(\xi_1)+a_1\sinh(\xi_2+\theta_2)}+\frac{f'(t)y^2}{4p}, \tag{2.22}$$

where

$$\theta_2=\ln\Big(-\frac{p_1a_1}{2a_2}\Big).$$

Case 3:

$$\begin{cases} a_2=-ia_1,\ a_1=a_1,\ p_1=p_1,\ p_2=p_2, \\ c_1=4p^2a_1^3,\ c_2=-4ip^2a_1^3. \end{cases} \tag{2.23}$$

Combining (2.13), (2.23), (2.11) with (2.1), we have

$$u=2pa_1\frac{p_1\sin(\xi_1)-i\exp(i\xi_1)+ip_2\exp(-i\xi_1)}{p_1\cos(\xi_1)+p_2\exp(-i\xi_1)+\exp(i\xi_1)}+\frac{f'(t)y^2}{4p}, \tag{2.24}$$

where

$$\xi_1=a_1(px+f(t)y)+4p^2a_1^3\int f(t)dt.$$

We make the dependent variable transformation in Eq. (2.24) as follows

$$a_1=iA_1, \tag{2.25}$$

where A_1 is real.

We obtain new form for Eq. (2.24) as follows

$$u=-2pA_1\frac{p_1\sinh(\xi_1^*)+p_2\exp(\xi_1^*)-\exp(-\xi_1^*)}{p_1\cosh(\xi_1^*)+p_2\exp(\xi_1^*)+\exp(-\xi_1^*)}+\frac{f'(t)y^2}{4p}, \tag{2.26}$$

where

$$\xi_1^*=A_1(px+f(t)y)-4p^2A_1^3\int f(t)dt.$$

Let $p_2>0$, then (2.28) can be written as follows

$$u_5=-2pA_1\frac{p_1\sinh(\xi_1^*)+2\sqrt{p}_2\sinh(\xi_1^*+\theta_3)}{p_1\cosh(\xi_1^*)+2\sqrt{p}_2\cosh(\xi_1^*+\theta_3)}+\frac{f'(t)y^2}{4p}, \tag{2.27}$$

where

$$\theta_3=\frac{1}{2}\ln p_2.$$

Let $p_2<0$, then (2.28) can be written as follows

$$u_6=2pA_1\frac{p_1\sinh(\xi_1^*)-2\sqrt{-p_2}\cosh(\xi_1^*+\theta_4)}{p_1\cosh(\xi_1)-2\sqrt{-p_2}\sinh(\xi_1^*+\theta_4)}+\frac{f'(t)y^2}{4p}, \tag{2.28}$$

where

$$\theta_4=\frac{1}{2}\ln(-p_2).$$

Case 4:

$$\begin{cases} a_1=ia_2,\ a_2=a_2,\ p_1=p_1,\ p_2=\dfrac{p_1^2}{4}, \\ c_1=-i(8p^2a_2^3+c_2),\ c_2=c_2. \end{cases} \tag{2.29}$$

Combining (2.13), (2.29), (2.11) with (2.1), we have

$$u=-2p\frac{p_1\sinh(\xi_1)+\frac{p_1^2}{4}\exp(\xi_2)-\exp(-\xi_2)}{p_1\cosh(\xi_1)+\frac{p_1^2}{4}\exp(\xi_2)+\exp(-\xi_2)}+\frac{f'(t)y^2}{4p}, \tag{2.30}$$

where

$$\xi_1=a_2(px+f(t)y)-(8p^2a_2^3+c_2)\int f(t)dt,$$
$$\xi_2=a_2(px+f(t)y)+c_2\int f(t)dt.$$

Let $p_1>0$, then (2.30) can be written as follows

$$u_7=-2p\frac{\sinh(\xi_1)+\sinh(\xi_2+\ln(\frac{p_1}{2}))}{\cosh(\xi_1)+\cosh(\xi_2+\ln(\frac{p_1}{2}))}+\frac{f'(t)y^2}{4p}. \tag{2.31}$$

Let $p_1<0$, then (2.30) can be written as follows

$$u_8=-2p\frac{\sinh(\xi_1)-\sinh\Big(\xi_2+\ln\big(-\frac{p_1}{2}\big)\Big)}{\cosh(\xi_1)-\cosh\big(\xi_2+\ln\big(-\frac{p_1}{2}\big)\big)}+\frac{f'(t)y^2}{4p}. \tag{2.32}$$

Then we look for a separable solution of the additive form

$$\theta=f(t)+g(y), \tag{2.33}$$

where $f(t)$, $g(y)$ are smooth functions to be determined later. Substituting (2.33) into (2.3), we have

$$q(y,t)=\frac{f'(t)y}{2p}. \tag{2.34}$$

Combining (2.33), (2.34) with (2.21), we obtain

$$\varphi_{\xi t}-6pg'(y)\varphi_{\xi}\varphi\xi\xi+p^2g'(y)\varphi_{\xi\xi\xi\xi}=0. \tag{2.35}$$

Let $g'(y)=$constant, then we assume without loss of generality that $g(y)=y$.

Integrating (2.35) once with respect to ξ, we obtain

$$\varphi_t - 3p\varphi_\xi^2 + p^2\varphi_{\xi\xi\xi} = 0. \tag{2.36}$$

In a manner similar to that of above we obtain

$$u_1 = -2pa_2\tanh\Big(\xi + \frac{1}{2}\ln p_2\Big) + \frac{f'(t)y}{2p}, \tag{2.37a}$$

$$u_2 = -2pa_2\coth\Big(\xi + \frac{1}{2}\ln(-p_2)\Big) + \frac{f'(t)y}{2p}, \tag{2.37b}$$

where

$$\xi = a_2(px+y+f(t)) - 4p^2a_2^3\int f(t)dt,$$

and

$$u_3 = 2a_1a_2p\frac{\sin(\xi_1)+\cosh(\xi_2+\theta_1)}{a_2\cos(\xi_1)-a_1\sinh(\xi_2+\theta_1)} + \frac{f'(t)y}{2p}, \tag{2.38a}$$

$$u_4 = 2a_1a_2p\frac{\sin(\xi_1)-\cosh(\xi_2+\theta_2)}{a_2\cos(\xi_1)+a_1\sinh(\xi_2+\theta_2)} + \frac{f'(t)y}{2p}, \tag{2.38b}$$

where

$$\xi_1 = a_1(px+y+f(t)) + a_1(p^2a_1^2 - 3p^2a_2^2)\int f(t)dt, \tag{2.39a}$$

$$\xi_2 = a_2(px+y+f(t)) + a_2(3p^2a_1^2 - p^2a_2^2)\int f(t)dt, \tag{2.39b}$$

$$u_5 = -2pA_1\frac{p_1\sinh(\xi_1^*)+2\sqrt{p_2}\sinh(\xi_1^*+\theta_3)}{p_1\cosh(\xi_1^*)+2\sqrt{p_2}\cosh(\xi_1^*+\theta_3)} + \frac{f'(t)y}{2p}, \tag{2.39c}$$

$$u_6 = 2pA_1\frac{p_1\sinh(\xi_1^*)-2\sqrt{-p_2}\cosh(\xi_1^*+\theta_4)}{p_1\cosh(\xi_1)-2\sqrt{-p_2}\sinh(\xi_1^*+\theta_4)} + \frac{f'(t)y}{2p}, \tag{2.39d}$$

where

$$\xi_1 = A_1(px+y+f(t)) - 4p^2A_1^3\int f(t)dt,$$

and

$$u_7 = -2p\frac{\sinh(\xi_1)+\sinh(\xi_2+\ln(\frac{p_1}{2}))}{\cosh(\xi_1)+\cosh(\xi_2+\ln(\frac{p_1}{2}))} + \frac{f'(t)y}{2p}, \tag{2.40a}$$

$$u_8 = -2p\frac{\sinh(\xi_1)-\sinh(\xi_2+\ln(-\frac{p_1}{2}))}{\cosh(\xi_1)-\cosh(\xi_2+\ln(-\frac{p_1}{2}))} + \frac{f'(t)y}{2p}, \tag{2.40b}$$

where

$$\xi_1 = a_2(px+y+f(t)) - (8p^2a_2^3 + c_2)\int f(t)dt,$$

$$\xi_2 = a_2(px+y+f(t)) + c_2\int f(t)dt.$$

For a function $u(x,y,t)$ of the form

$$u(x,y,t)=v(x,\tau),\tau=\psi(t+k^2y). \tag{2.41}$$

Eq. (1.1) becomes

$$\psi' v_{x\tau}-4k^2\psi' v_x v_{x\tau}-2k^2\psi' v_\tau v_{xx}+k^2\psi' v_{xxx\tau}=0. \tag{2.42}$$

Eliminating ψ', we have

$$v_{x\tau}-4k^2 v_x v_{x\tau}-2k^2 v_\tau v_{xx}+k^2 v_{xxx\tau}=0, \tag{2.43}$$

which, like the KdV equation, is an integrable model equation for the propagation of long waves in a medium with nonlinear dispersion [27,28].

To solve Eq. (2.43), we make the dependent variable transformation

$$v=-2(\ln f)_x, \tag{2.44}$$

where f is an unknown real function which will be determined.

Substituting (2.44) into (2.43), we obtain

$$(\ln f)_{xx\tau}+8k^2(\ln f)_{xx}(\ln f)_{xx\tau}+4k^2(\ln f)_{x\tau}(\ln f)_{xxx}+k^2(\ln f)_{xxxx\tau}=0. \tag{2.45}$$

Integrating (2.45) once with respect to x, we have

$$\begin{aligned}&(\ln f)_{x\tau}+6k^2(\ln f)_{xx}(\ln f)_{x\tau}+k^2(\ln f)_{xxx\tau}\\&\quad+2k^2\partial_x^{-1}[(\ln f)_{xx}(\ln f)_{xx\tau}-(\ln f)_{x\tau}(\ln f)_{xxx}]=C.\end{aligned} \tag{2.46}$$

Taking $C=0$, therefore (2.46) can be written as

$$(D_\tau D_x+k^2D_\tau D_x^3)f\cdot f+4k^2f^2\partial_x^{-1}(D_x(\ln f)_{x\tau}\cdot(\ln f)_{xx})=0. \tag{2.47}$$

Suppose that

$$\partial_x^{-1}(D_x(\ln f)_{x\tau}\cdot(\ln f)_{xx})=0, \tag{2.48}$$

note that to have a correct solution for Eq. (2.43) we must consider (2.48) in our algebraic systems of equations. Therefore, by our assumption, Eq. (2.47) reduces to

$$(D_\tau D_x+k^2D_\tau D_x^3)f\cdot f=0. \tag{2.49}$$

Now we suppose that the solution of Eq. (2.47) as

$$f=p_1\cos(\xi_1)+p_2\exp(\xi_2)+\exp(-\xi_2), \tag{2.50}$$

where $\xi_i=a_ix+b_i\tau$, $(i=1,2)$. a_i, b_i are some constants to be determined later.

Substituting (2.50) into (2.49) and equating all coefficients of $\exp(\xi_2)\cos(\xi_1)$, $\exp(-\xi_2)\cos(\xi_1)$, $\exp(\xi_2)\sin(\xi_1)$, $\exp(-\xi_2)\sin(\xi_1)$ to zero, we get a set of nonlinear algebra equations for a_i, b_i, p_i, $(i=1,2)$,

$$\begin{cases} a_1b_1-a_2b_2-k^2a_1^3b_1-k^2a_2^3b_2+3k^2a_1a_2^2b_1+3k^2a_1^2a_2b_2=0, \\ a_1b_2+a_2b_1-k^2a_1^3b_2+k^2a_2^3b_1+3k^2a_1a_2^2b_2-3k^2a_1^2a_2b_1=0, \\ p_1^2a_1b_1-4p_2a_2b_2-4k^2p_1^2a_1^3b_1-16k^2p_2a_2^3b_2=0. \end{cases} \tag{2.51}$$

Substituting (2.50) into (2.48) and equating all coefficients of $\sin(\xi_1)$, $\exp(\xi_2)$, $\exp(-\xi_2)$ to zero, we obtain a set of nonlinear algebra equations for a_i, b_i, $(i=1,2)$,

$$\begin{cases} a_2^3b_1+a_1^2a_2b_1-a_1^3b_2-a_1a_2^2b_2=0, \\ a_1^3b_2+a_1a_2^2b_2-a_1^2a_2b_1-a_2^3b_1=0. \end{cases} \tag{2.52}$$

Solving the system of Eqs. (2.51) and (2.52) with the aid of Maple, one gets the following cases:

Case 1:

$$a_1=0, \quad a_2=\frac{i}{k}, \quad b_1=b_1, \quad b_2=b_2, \quad p_1=p_1, \quad p_2=0. \tag{2.53}$$

Substituting (2.53) into (2.50) yields

$$f=p_1\cosh b_1\tau+\exp\Big(-\frac{i}{k}x+b_2\tau\Big). \tag{2.54}$$

From (2.54), (2.44) and (2.41), we obtain the exact solutions to the Calogero Equation (1.1) given that

$$u=-\frac{2i}{k}\frac{\exp[-(\frac{i}{k}x+b_2\psi(k^2y+t))]}{p_1\cosh(b_1\psi(k^2y+t))+\exp[-(\frac{i}{k}x+b_2\psi(k^2y+t))]}, \tag{2.55}$$

where b_1, b_2, k are arbitrary constants.

We make the dependent variable transformation in Eq. (2.55) as follows

$$k=iK,$$

where K is real. We obtain new form for Eq. (2.55) as follows

$$u=-\frac{2}{K}\frac{\exp[-(\frac{x}{K}+b_2\psi(t-K^2y))]}{p_1\cosh(b_1\psi(t-K^2y))+\exp[-(\frac{x}{K}+b_2\psi(t-K^2y))]}. \tag{2.56}$$

Case 2:

$$a_1=\frac{ib_1}{2kb_2}, \quad a_2=\frac{i}{2k}, \quad b_1=b_1, \quad b_2=b_2, \quad p_1=0, \quad p_2=p_2. \tag{2.57}$$

Substituting (2.57) into (2.50) yields

$$f=p_2\exp(\xi_2)+\exp(-\xi_2). \tag{2.58}$$

From (2.58), (2.44) and (2.41), one obtains

$$u=-\frac{i}{k}\frac{p_2\exp(\xi_2)-\exp(-\xi_2)}{p_2\exp(\xi_2)+\exp(-\xi_2)}, \tag{2.59}$$

where

$$\xi_2=\frac{i}{2k}x+b_2\psi(k^2y+t),$$

b_2, k are arbitrary constants.

We make the dependent variable transformation in Eq. (2.59) as follows

$$k=iK,$$

where K is real. We obtain new form for Eq. (2.59) as follows

$$u=-\frac{1}{K}\frac{p_2\exp(\xi_2^*)-\exp(-\xi_2^*)}{p_2\exp(\xi_2^*)+\exp(-\xi_2^*)}, \tag{2.60}$$

where

$$\xi_2^*=\frac{x}{2K}+b_2\psi(t-K^2y),$$

b_2, K are arbitrary constants.

If $p_2>0$, taking $\theta_1=\frac{1}{2}\ln p_2$, then we obtain

$$u=-\frac{1}{K}\tanh(\xi_2^*+\theta_1). \tag{2.61}$$

If $p_2<0$, taking $\theta_2=\ln(\sqrt{-p_2})$, then we obtain

$$u=-\frac{1}{K}\coth(\xi_2^*+\theta_2). \tag{2.62}$$

Case 3:

$$a_1=ia_2,\quad a_2=a_2,\quad b_1=-ib_2,\quad b_2=b_2,\quad p_2=\frac{1}{4}p_1^2,\quad p_1=p_1. \tag{2.63}$$

Substituting (2.63) into (2.50) yields

$$f=p_1\cosh(\xi_1)+\frac{1}{4}p_1^2\exp(\xi_2)+\exp(-\xi_2), \tag{2.64}$$

where $\xi_1=a_2x-b_2\psi(k^2y+t)$, $\xi_2=a_2x+b_2\psi(k^2y+t)$, k, a_2, b_2 are arbitrary constants.

Combining (2.64),(2.44) with (2.41), one gets

$$u=-2a_2\frac{p_1\sinh(\xi_1)+\frac{1}{4}p_1^2\exp(\xi_2)-\exp(-\xi_2)}{p_1\cosh(\xi_1)+\frac{1}{4}p_1^2\exp(\xi_2)+\exp(-\xi_2)}. \tag{2.65}$$

If $p_1>0$, taking $\theta_3=\ln\frac{p_1}{2}$, then we obtain

$$u=-2a_2\frac{\sinh(\xi_1)+\sinh(\xi_2+\theta_3)}{\cosh(\xi_1)+\cosh(\xi_2+\theta_3)}. \tag{2.66}$$

If $p_1<0$, taking $\theta_4=\ln(-\frac{p_1}{2})$, then we obtain

$$u=-2a_2\frac{\sinh(\xi_1)-\sinh(\xi_2+\theta_4)}{\cosh(\xi_1)-\cosh(\xi_2+\theta_4)}. \tag{2.67}$$

Case 4:

$$a_1=\frac{1}{2k},\quad a_2=\frac{i}{2k},\quad b_1=b_1,\quad b_2=b_2,\quad p_1=p_1,\quad p_2=p_2. \tag{2.68}$$

Substituting (2.68) into (2.50) yields

$$f=p_1\cos(\xi_1)+p_2\exp(\xi_2)+\exp(-\xi_2). \tag{2.69}$$

Combining (2.69), (2.44) with (2.41), we obtain

$$u=-\frac{1}{k}\frac{-p_1\sin(\xi_1)+ip_2\exp(\xi_2)-i\exp(-\xi_2)}{p_1\cos(\xi_1)+p_2\exp(\xi_2)+\exp(-\xi_2)}, \tag{2.70}$$

where

$$\xi_1=\frac{1}{2k}x+b_1\psi(k^2y+t),\quad \xi_2=\frac{i}{2k}x+b_2\psi(k^2y+t),$$

k, a_2, b_2 are arbitrary constants.

We make the dependent variable transformation in Eq. (2.70) as follows

$$k=iK,\quad b_1=iB_1,$$

where K, B_1 are real. We obtain new form for Eq. (2.70) as follows

$$u=\frac{1}{K}\frac{-p_1\sinh(\xi_1^*)-p_2\exp(\xi_2^*)+\exp(-\xi_2^*)}{p_1\cosh(\xi_1^*)+p_2\exp(\xi_2^*)+\exp(-\xi_2^*)}, \tag{2.71}$$

where

$$\xi_1^*=\frac{x}{2K}-B_1\psi(t-K^2y),$$
$$\xi_2^*=\frac{x}{2K}+b_2\psi(t-K^2y).$$

If $p_2>0$, taking $\theta_5=\ln\sqrt{p_2}$, then we obtain

$$u=-\frac{1}{K}\frac{p_1\sinh(\xi_1^*)+2\sqrt{p_2}\sinh(\xi_2^*+\theta_5)}{p_1\cosh(\xi_1^*)+2\sqrt{p_2}\cosh(\xi_2^*+\theta_5)}. \tag{2.72}$$

If $p_2<0$, taking $\theta_6=\ln\sqrt{-p_2}$, then we obtain

$$u=\frac{1}{K}\frac{-p_1\sinh(\xi_1^*)+2\sqrt{-p_2}\cosh(\xi_2^*+\theta_6)}{p_1\cosh(\xi_1^*)-2\sqrt{-p_2}\sinh(\xi_2^*+\theta_6)}. \tag{2.73}$$

Case 5:

$$a_1=-\frac{1}{2k},\quad a_2=\frac{i}{2k},\quad b_1=b_1,\quad b_2=b_2,\quad p_1=p_1,\quad p_2=p_2. \tag{2.74}$$

Combining (2.74) with (2.50), one gets

$$f=p_1\cos(\xi_1)+p_2\exp(\xi_2)+\exp(-\xi_2), \tag{2.75}$$

where

$$\xi_1=-\frac{1}{2k}x+b_1\tau,\quad \xi_2=\frac{i}{2k}x+b_2\tau,$$

a_2, b_2 are arbitrary constants.

Combining (2.75), (2.44) with (2.41) yields

$$u=-\frac{1}{k}\frac{p_1\sin(\xi_1)+ip_2\exp(\xi_2)-i\exp(-\xi_2)}{p_1\cos(\xi_1)+p_2\exp(\xi_2)+\exp(-\xi_2)}, \tag{2.76}$$

where

$$\xi_1=-\frac{1}{2k}x+b_1\psi(k^2y+t),\quad \xi_2=\frac{i}{2k}x+b_2\psi(k^2y+t),$$

a_2, b_2 are arbitrary constants.

We make the dependent variable transformation in Eq. (2.76) as follows

$$k=iK,\quad b_1=iB_1,$$

where K, B_1 are real. We obtain new form for Eq. (2.76) as follows

$$u=\frac{1}{K}\frac{-p_1\sinh(\xi_1^*)-p_2\exp(\xi_2^*)+\exp(-\xi_2^*)}{p_1\cosh(\xi_1^*)+p_2\exp(\xi_2^*)+\exp(-\xi_2^*)}, \tag{2.77}$$

where

$$\xi_1^*=\frac{x}{2K}+B_1\psi(t-K^2y),$$
$$\xi_2^*=\frac{x}{2K}+b_2\psi(t-K^2y).$$

If $p_2>0$, taking $\theta_7=\ln\sqrt{p_2}$, then we obtain

$$u=-\frac{1}{K}\frac{p_1\sinh(\xi_1^*)+2\sqrt{p_2}\sinh(\xi_2^*+\theta_7)}{p_1\cosh(\xi_1^*)+2\sqrt{p_2}\cosh(\xi_2^*+\theta_7)}. \tag{2.78}$$

If $p_2<0$, taking $\theta_8=\ln\sqrt{-p_2}$, then we obtain

$$u=\frac{1}{K}\frac{-p_1\sinh(\xi_1^*)+2\sqrt{-p_2}\cosh(\xi_2^*+\theta_8)}{p_1\cosh(\xi_1^*)-2\sqrt{-p_2}\sinh(\xi_2^*+\theta_8)}. \tag{2.79}$$

3 Discussion

It is obvious to see that the solutions above recover the solutions u_1-u_5 in [21]. And the solutions (2.38a), (2.38b), (2.40a), (2.40b), (2.56), (2.66), (2.67), (2.72), (2.78) cannot be obtained in [21]. When taking the arbitrary function $f(t)$, ψ as special constants or functions, we can derive rich exact solutions for Eq. (1.1).

4 Conclusions

In this paper, the idea of a combination of variable separation approach and the extended homoclinic test approach is proposed to seek non-travelling wave solutions of Calogero equation. The equation is reduced to some $(1+1)$-dimensional nonlinear equations by applying the variable separation approach and solves reduced equations with the extended homoclinic test technique. Based on this idea and with the aid of symbolic computation, some new explicit solutions can be obtained.

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References

[1] R. ZHDANOV, *Separation of variables in the nonlinear wave equation,* J. Phys. A, 27 (1994), pp. L291–L297.

[2] S. LOU AND L. CHEN, *Formal variable separation approach for nonintegrable models,* J. Math. Phys., 40 (1999), pp. 6491–6500.

[3] P. G. ESTEVEZ AND C. QU, *Separation of variables in a nonlinear wave equation with a variable wave speed,* Theoret. Math. Phys., 133 (2002), pp. 1490–1497.

[4] S. ZHANG, S. LOU AND C. QU, *Variable separation and exact solutions to generalized nonlinear diffusion equations,* Chin. Phys. Lett., 19 (2002), pp. 1741–1744.

[5] S. ZHANG, S. LOU AND C. QU, *New variable separation approach: application to nonlinear diffusion equations,* J. Phys. A, 36 (2003), pp. 12223–12242.

[6] S. ZHANG AND S. LOU, *Derivative-dependent functional separable solutions for the KdV-type equations,* Phys. A, 335 (2004), pp. 430–444.

[7] S. LOU AND J. LU, *Special solutions from the variable separation approach: the Davey-Stewartson equation,* J. Phys. A, 29 (1996), pp. 4209–4215.

[8] J. YING AND S. LOU, *Multilinear variable separation approach in $(3+1)$-dimensions: the Burgers equation,* Chin. phys. Lett., 20 (2003), pp. 1448.

[9] X. TANG AND S. LOU, *A variable separation approach to solve the integrable and nonintegrable models: coherent structures of the $(2+1)$-dimensional KdV equation*, Commun. Theor. Phys., 38 (2002), pp. 1–8.

[10] F. CALOGERO AND A. DEGASPERIS, *Nonlinear evolution equations solvable by the inverse spectral transform I,* Nuovo Cimento B, (11)(32) (1976), pp. 201–242.

[11] F. CALOGERO AND A. DEGASPERIS, *Nonlinear evolution equations solvable by the inverse spectral transform II,* Nuovo Cimento B, (11)(39) (1977), pp. 1–54.

[12] O. BOGOYAVLENSKII, *Overturning solitons in new two-dimensional integrable equations*, Math. USSR-Izv., 34 (1990), pp. 245–259.

[13] O. BOGOYAVLENSKII, *Breaking solitons III,* Math. USSR-Izv., 36 (1991), pp. 129–137.

[14] J. SCHIFF, Painlevé Transendent, Their Asymptotics and Physical Applications, Plenum, New York, 1992, pp. 393.

[15] A. BEKIR, *Painlevé test for some* $(2+1)$*-dimensional nonlinear equations,* Chaos Solitons Fractals, 32 (2007), pp. 449–455.

[16] E. FAN AND K. CHOW, *Darboux covariant Lax pairs and infinite conservation laws of the* $(2+1)$*-dimensional breaking soliton equation,* J. Math. Phys., 52 (2011), pp. 1–10.

[17] T. ALAGESAN, Y. CHUNG AND K. NAKKEERAN, *Painlevé test for the certain* $(2+1)$*-dimensional nonlinear evolution equations,* Chaos Solitons Fractals, 26 (2005), pp. 1203–1209.

[18] X. YONG, Z. ZHANG AND Y. CHEN, *Bäcklund transformation, nonlinear superposition formula and solutions of the Calogero equation,* Phys. Lett. A, 372 (2008), pp. 6273–6279.

[19] Y. GAO AND B. TIAN, *New family of overturning soliton solutions for a typical breaking soliton equation,* Comput. Math. Appl., 30 (1995), pp. 97–100.

[20] Z. YAN AND H. ZHANG, *Constructing families of soliton-like solutions to a* $(2+1)$*-dimensional breaking soliton equation using symbolic computation,* Comput. Math. Appl., 44 (2002), pp. 1439–1444.

[21] D. XIAN, *Symmetry reduction and new non-traveling wave solutions of* $(2+1)$*-dimensional breaking soliton equation,* Commun. Nonlinear Sci. Numer. Simul., 15 (2010), pp. 2061–2065.

[22] B. TIAN, K. ZHAO AND Y. GAO, *Symbolic computation in engineering: application to a breaking soliton equation,* Internat. J. Eng. Sci., 35 (1997), pp. 1081–1083.

[23] E. FAN AND H. ZHANG, *A note on the homogeneous balance method,* Phys. Lett. A, 246 (1998), pp. 403–406.

[24] Z. LÜ, L. DUAN AND F. XIE, *Cross soliton-like waves for the* $(2+1)$*-dimensional breaking soliton equation,* Chin. Phys. Lett., 27 (2010), pp. 1–3.

[25] Q. FENG AND B. ZHENG, *Exact traveling wave solution for the* $(2+1)$ *dimensional breaking Soliton equation,* Proceedings of the 2010 American Conference on Applied Mathematics, WSEAS, 2010, pp. 440–442.

[26] Z. ZHAO, Z. DAI AND S. HAN, *The EHTA for nonlinear evolution equations,* Appl. Math. Comput., 217 (2010), pp. 4306–4310.

[27] R. HIROTA AND J. SATSUMA, *N-soliton solutions of model equations for shallow water waves,* J. Phys. Soc. Japan, 40 (1976), pp. 611–612.

[28] P. CLARKSON AND E. L. MANSFIELD, *On a shallow water wave equation,* Nonlinearity, 7 (1997), pp. 975–1000.