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Restricted Additive Schwarz Preconditioner for Elliptic Equations with Jump Coefficients

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Abstract. This paper provides a proof of robustness of the restricted additive Schwarz preconditioner with harmonic overlap (RASHO) for the second order elliptic problems with jump coefficients. By analyzing the eigenvalue distribution of the RASHO preconditioner, we prove that the convergence rate of preconditioned conjugate gradient method with RASHO preconditioner is uniform with respect to the large jump and meshsize.

AMS subject classifications: 65N30, 65N55, 65F10

Key words: Jump coefficients, conjugate gradient, effective condition number, domain decomposition, restricted additive Schwarz.

1 Introduction

In this paper, we will discuss the restricted additive Schwarz (with harmonic overlap) preconditioned conjugate gradient method for the linear finite element approximation of the second order elliptic boundary value problem

$$\begin{cases} -\nabla\cdot(\omega\nabla u)=f & \text{in } \Omega, \\ u=g_D & \text{on } \Gamma_D, \\ -\omega\dfrac{\partial u}{\partial n}=g_N & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\Omega\in R^d$ ($d=1,2$ or 3) is a polygonal or polyhedral domain with Dirichlet boundary Γ_D and Neumann boundary Γ_N. The coefficient $\omega=\omega(x)$ is a positive and piecewise

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constant function. More precisely, we assume that there are M open disjointed polygonal or polyhedral regions Ω_m $(m=1,\cdots,M)$ satisfying $\cup_{m=1}^{M}\overline{\Omega}_m=\overline{\Omega}$ with

$$\omega_m=\omega|_{\Omega_m}, \quad m=1,\cdots,M,$$

where each $\omega_m>0$ is a constant. The analysis can be carried through to a more general case when $\omega(x)$ varies moderately in each subdomain.

The goal of this paper is to prove the robustness of restricted additive Schwarz preconditioner with harmonic overlap (RASHO-PCG). The design of RASHO method is based on a much deeper understanding of the behavior of Schwarz-type methods (see [4] and the references cited therein). It is a modification and symmetrized version of RAS (introduced in [3]). In this paper, we solve numerically the second elliptic problem with jump coefficients by RASHO-PCG method. As a result, we prove that the convergence rate of RASHO-PCG method is uniform with respect to the large jump and meshsize. We will see that the effective condition number of RASHO-PCG is $C|\log H|\cdot r^2$.

The rest of the paper is organized as follows. In order to make the paper be self-contained, we refer directly to parts of contents in [9] and [10]. In Section 2, we introduce some basic notation, the PCG algorithm and some theoretical foundations. In Section 3, we quote some main results on the weighted L^2-projection from [2]. Section 4 is an introduction of the RASHO preconditioner. In Section 5, we analyze the eigenvalue distribution of the RASHO preconditioned system and prove the convergence rate of the PCG algorithm. In Section 6, we give some conclusion remarks.

2 Preliminaries

2.1 Notation

We introduce the bilinear form

$$a(u,v)=\sum_{m=1}^{M}\omega_m(\nabla u,\nabla v)_{L^2(\Omega_m)}, \quad \forall u,v\in H_D^1(\Omega),$$

where $H_D^1(\Omega)=\{v\in H^1(\Omega): v|_{\Gamma_D}=0\}$, and introduce the H^1-norm and seminorm with respect to any subregion Ω_m by

$$|u|_{1,\Omega_m}=\|\nabla u\|_{0,\Omega_m}, \quad \|u\|_{1,\Omega_m}=(\|u\|_{0,\Omega_m}^2+|u|_{1,\Omega_m}^2)^{\frac{1}{2}}.$$

Thus,

$$a(u,u)=\sum_{m=1}^{M}\omega_m|u|_{1,\Omega_m}^2:=|u|_{1,\omega}^2.$$

We also need the weighted L^2-inner product

$$(u,v)_{0,\omega}=\sum_{m=1}^{M}\omega_m(u,v)_{L^2(\Omega_m)}$$

and the weighted L^2- and H^1-norms

$$\|u\|_{0,\omega} = (u,u)_{0,\omega}^{\frac{1}{2}}, \quad \|u\|_{1,\omega} = (\|u\|_{0,\omega}^2 + |u|_{1,\omega}^2)^{\frac{1}{2}}.$$

For any subset $O \subset \Omega$, let $|u|_{1,\omega,O}$ and $\|u\|_{0,\omega,O}$ be the restrictions of $|u|_{1,\omega}$ and $\|u\|_{0,\omega}$ on the subset O, respectively.

For the distribution of the coefficients, we introduce the index set

$$I = \{m : \text{meas}(\partial\Omega_m \cap \Gamma_D) = 0\},$$

where meas$(\cdot)$ is the $d-1$ measure. In other words, I is the index set of all subregions which do not touch the Dirichlet boundary. We assume that the cardinality of I is m_0. We shall emphasize that m_0 is a constant which depends only on the distribution of the coefficients.

2.2 The discrete system

Given a quasi-uniform triangulation $\mathcal{T}_h$ with the meshsize h, let

$$\mathcal{V}_h = \{v \in H_D^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau),\ \forall \tau \in \mathcal{T}_h\}$$

be the piecewise linear finite element space, where $\mathcal{P}_1$ denotes the set of linear polynomials. The finite element approximation of (1.1) is the function $u \in \mathcal{V}_h$, such that

$$a(u,v) = (f,v) + \int_{\Gamma_N} g_N v, \quad \forall v \in \mathcal{V}_h.$$

We define a linear symmetric positive definite operator $A : \mathcal{V}_h \to \mathcal{V}_h$ by

$$(Au,v)_{0,\omega} = a(u,v).$$

The related inner product and the induced energy norm are denoted by

$$(\cdot,\cdot)_A := a(\cdot,\cdot), \quad \|\cdot\|_A := \sqrt{a(\cdot,\cdot)}.$$

Then we have the following operator equation,

$$Au = F. \tag{2.1}$$

2.3 Preconditioned Conjugate Gradient (PCG) methods

The well known conjugate gradient method is the basis of all the preconditioning techniques to be studied in this paper. The PCG methods can be viewed as a conjugate gradient method applied to the preconditioned system

$$BAu = BF.$$

Here, B is an SPD operator, known as a preconditioner of A. Note that BA is symmetric with respect to the inner product $(\cdot,\cdot)_{B^{-1}}$ (or $(\cdot,\cdot)_A$). For the implementation of the PCG algorithm, we refer to the monographs [1,6,7].

Let u_k, $k=0,1,\cdots$, be the solution sequence of the PCG algorithm. It is well known that

$$\|u-u_k\|_A \le 2\Big(\frac{\sqrt{k(BA)}-1}{\sqrt{k(BA)}+1}\Big)^k \|u-u_0\|_A, \tag{2.2}$$

which implies that the PCG method generally converges faster with a smaller condition number $k(BA)$.

Even though the estimate given in (2.2) is sufficient for many applications, in general it is not sharp. One way to improve the estimate is to look at the eigenvalue distribution of BA(see [1,5] for more details). More specifically, suppose that we can divide $\sigma(BA)$, the spectrum of BA, into two sets, $\sigma_0(BA)$ and $\sigma_1(BA)$, where σ_0 consists of all "bad" eigenvalues and the remaining eigenvalues in σ_1 are bounded above and below, then we have the following theorem.

Theorem 2.1. *Suppose that $\sigma(BA)=\sigma_0(BA)\cup\sigma_1(BA)$ such that there are m elements in $\sigma_0(BA)$ and $\lambda\in[a,b]$ for each $\lambda\in\sigma_1(BA)$. Then*

$$\|u-u_k\|_A \le 2K\Big(\frac{\sqrt{b/a}-1}{\sqrt{b/a}+1}\Big)^{k-m} \|u-u_0\|_A, \tag{2.3}$$

where

$$K=\max_{\lambda\in\sigma_1(BA)}\prod_{\mu\in\sigma_0(BA)}\Big|1-\frac{\lambda}{\mu}\Big|.$$

If there are only m small eigenvalues in σ_0, say

$$0<\lambda_1\le\lambda_2\cdots\le\lambda_m\ll\lambda_{m+1}\le\cdots\le\lambda_n,$$

then

$$K=\prod_{i=1}^m\Big|1-\frac{\lambda_n}{\lambda_i}\Big|\le\Big(\frac{\lambda_n}{\lambda_1}-1\Big)^m=(k(BA)-1)^m.$$

In this case, the convergence rate estimate (2.3) *becomes*

$$\frac{\|u-u_k\|_A}{\|u-u_0\|_A}\le 2(k(BA)-1)^m\Big(\frac{\sqrt{b/a}-1}{\sqrt{b/a}+1}\Big)^{k-m}. \tag{2.4}$$

Based on (5.4)*, given a tolerance $0<\epsilon<1$, the number of iterations of the PCG algorithm needed for*

$$\frac{\|u-u_k\|_A}{\|u-u_0\|_A}<\epsilon$$

is given by

$$k\ge m+\Big(\log\Big(\frac{2}{\epsilon}\Big)+m\log(k(BA)-1)\Big)\Big/\log\Big(\frac{\sqrt{b/a}+1}{\sqrt{b/a}-1}\Big). \tag{2.5}$$

Observing the convergent estimate (5.4), if there are only a few small eigenvalues of BA in $\sigma_0(BA)$, then the convergent rate of the PCG methods will be dominated by the factor $\frac{\sqrt{b/a}+1}{\sqrt{b/a}-1}$, i.e., by b/a, where $b=\lambda_n(BA)$ and $a=\lambda_{m+1}(BA)$. We define this quantity as the "effective condition number".

Definition 2.1 (see [9])**.** Let $\mathcal{V}$ be a Hilbert space. The m-th effective condition number of an operator $A:\mathcal{V}\to\mathcal{V}$ is defined by

$$k_{m+1}(A)=\frac{\lambda_{\max}(A)}{\lambda_{m+1}(A)},$$

where $\lambda_{m+1}(A)$ is the $(m+1)$-th minimal eigenvalue of A.

To estimate the effective condition number, we need to estimate $\lambda_{m+1}(A)$. A fundamental tool is the following

Theorem 2.2 (Courant-Fisher min-max Theorem)**.** *The eigenvalues of a SPD operator $A:\mathcal{V}\to\mathcal{V}$ are characterized by the relation*

$$\lambda_{m+1}(A)=\min_{S,\ dim(S)=n-m}\ \max_{x\in S,\ x\neq 0}\frac{(Ax,x)}{(x,x)}. \tag{2.6}$$

Especially, for any subspace $\mathcal{V}_0\subset\mathcal{V}$ with $dim(\mathcal{V}_0)=n-m$, the following estimation of $\lambda_{m+1}(A)$ holds:

$$\lambda_{m+1}(A)\geq\min_{0\neq x\in\mathcal{V}_0}\frac{(Ax,x)}{(x,x)}. \tag{2.7}$$

3 Weighted L^2-projection

A major tool to analyze the RASHO preconditioner in this paper is the weighted L^2-projection $Q_H^{\omega}:L^2(\Omega)\to\mathcal{V}_H$ defined by

$$(Q_H^{\omega}u,v_H)_{0,\omega}=(u,v_H)_{0,\omega},\quad \forall v_H\in\mathcal{V}_H.$$

In this section, we shall recall two main results on weighted L^2-projection from [2]. And following [8], short notation $x\lesssim y$ means $x\leq Cy$.

Lemma 3.1 (see [2])**.** *Let $\mathcal{V}_H\subset\mathcal{V}_h$ be two nested linear finite element spaces. Then for any $u\in\mathcal{V}_h$, there holds,*

$$\|(I-Q_H^{\omega})u\|_{0,\omega}\leq c_d(h,H)H|u|_{1,\omega},$$

and

$$|Q_H^{\omega}u|_{1,\omega}\lesssim c_d(h,H)|u|_{1,\omega},$$

where

$$c_d(h,H)=C\cdot\begin{cases}\left(\dfrac{H}{h}\right)^{1/2}, & if\ d=3,\\ \left(\log\dfrac{H}{h}\right)^{1/2}, & if\ d=2.\end{cases}$$

Lemma 3.2 (see [10]). *For any*

$$u \in \widetilde{H}_D^1(\Omega) = \Big\{ u \in H_D^1(\Omega) : \int_{\Omega_m} u dx = 0, \ \forall m \in I \Big\},$$

we have

$$\|(I - Q_H^\omega) u\|_{0,\omega} \lesssim H |\log H|^{\frac{1}{2}} |u|_{1,\omega}$$

and

$$|Q_H^\omega u|_{1,\omega} \lesssim |\log H|^{\frac{1}{2}} |u|_{1,\omega}.$$

4 Restricted additive Schwarz preconditioners

In this section, we introduce the one- and two-level restricted additive Schwarz methods with harmonic overlap. Let $\mathcal{T}_h$ be fine finite element grid with meshsize h, on which the solution is sought. There is also a coarse grid $\mathcal{T}_H$ with meshsize H. For simplicity, we assume that each element in $\mathcal{T}_H$ is a union of some elements in $\mathcal{T}_h$, and also assume that $\mathcal{T}_H$ aligns with the jump interface.

4.1 Notation

We partition the domain Ω into N nonoverlapping subdomains Ω_i $(i = 1, \cdots, N)$, such that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega_i}$. Let n be the total number of interior nodes of $\mathcal{T}_h(\Omega)$, and W the set containing all the interior nodes. And let W_i^0 be total interior nodes of Ω_i, whose union is W.

We define the overlapping partition of W as follow. Let W_i^1 be the one-overlap partition of W, where $W_i^1 \supset W_i^0$ is obtained by including all the immediate neighboring vertices of all vertices in W_i^0. Using the idea recursively, we can define a δ-overlap partition of W,

$$W = \bigcup_{i=1}^N W_i^\delta.$$

Here the integer δ indicates the level of overlap with its neighboring subdomains, and δh is approximately the length of the extension.

We next define a subregion of Ω induced by a subset of nodes of $\mathcal{T}_h(\Omega)$ as follows. Let Z be a subset of W. The induced subregion, denoted by $\Omega(Z)$ is defined as the union of (1) the set Z itself, (2) the union of all the open elements (triangles) of $\mathcal{T}_h(\Omega)$ that have at least one vertex in Z, and (3) the union of the open edges of these triangles that have at least one endpoint as vertex of Z. Note that $\Omega(Z)$ is always an open region. The extended subregion Ω_i^δ is defined as $\Omega(W_i^\delta)$, and the corresponding subspace as

$$\mathcal{V}_i^\delta \equiv \mathcal{V} \cap H_0^1(\Omega_i^\delta) \quad \text{extended by zero to } \ \Omega \setminus \Omega_i^\delta.$$

It is easy to verify that

$$\mathcal{V}=\mathcal{V}_1^\delta+\mathcal{V}_2^\delta+\cdots+\mathcal{V}_N^\delta.$$

This decomposition is used in defining the classical one-level AS algorithm.

In the classical AS as defined above, all the nodes of W_i^δ are treated equally even through some subsets of the nodes play different roles in determining the convergence rate of the AS-preconditioned CG. To further understand the issue, we classify the nodes as follows. Let $\Gamma_i^\delta=\partial\Omega_i^\delta\backslash\partial\Omega$, i.e., the part of the boundary of Ω_i^δ that does not belong to the Dirichlet part of the physical boundary $\partial\Omega$. We define the interface-overlapping boundary Γ^δ as the union of all Γ_i^δ; i.e., $\Gamma^\delta=\bigcup_{i=1}^N\Gamma_i^\delta$. We also need to define the following subsets of W:

$$\begin{aligned}
&W^{\Gamma^\delta}\equiv W\bigcap\Gamma^\delta, && W_i^{\Gamma^\delta}\equiv W^{\Gamma^\delta}\bigcap W_i^\delta,\\
&W_{i,in}^{\Gamma^\delta}\equiv W^{\Gamma^\delta}\bigcap W_i^0, && W_{i,cut}^{\Gamma^\delta}\equiv W_i^{\Gamma^\delta}\backslash W_{i,in}^{\Gamma^\delta},\\
&W_{i,ovl}^\delta\equiv (W_i^\delta\backslash W_i^{\Gamma^\delta})\bigcap\Big(\bigcup_{j\neq i}W_j^\delta\Big), && W_{i,non}^\delta\equiv W_i^\delta\backslash\Big(W_i^{\Gamma^\delta}\bigcup W_{i,ovl}^\delta\Big),\\
&W_{i,in}^\delta\equiv W_{i,non}^\delta\bigcup W_{i,in}^{\Gamma^\delta}.
\end{aligned}$$

We frequently use functions that are discrete harmonic at certain nodes. Let $x\in W$ be a mesh point and $\phi_{x_k}(x)\in\mathcal{V}$ the finite element basis function associated with x_k; i.e., $\phi_{x_k}(x_k)=1$, and $\phi_{x_k}(x_j)=0$, $j\neq k$. We say that $u\in\mathcal{V}$ is discrete harmonic at x_k if

$$a(u,\phi_{x_k})=0.$$

If u is discrete harmonic at a set of nodal points Z, we say that u is discrete harmonic in $\Omega(Z)$.

New algorithm will be built on the subspace $\widetilde{\mathcal{V}}_i^\delta$ defined as a subspace of $\mathcal{V}_i^\delta$. $\widetilde{\mathcal{V}}_i^\delta$ consists of all functions that vanish on the cutting nodes $W_{i,cut}^{\Gamma^\delta}$ and are discrete harmonic at the nodes of $W_{i,ovl}^\delta$. Note that the degrees of freedom associated with the subspace $\widetilde{\mathcal{V}}_i^\delta$ are

$$\widetilde{W}_i^\delta\equiv W_i^\delta\backslash W_{i,cut}^{\Gamma^\delta}.$$

It is easy to see that $\Omega(\widetilde{W}_i^\delta)$ is the same as Ω_i^δ but with cuts. We denote $\Omega(\widetilde{W}_i^\delta)$ by $\widetilde{\Omega}_i^\delta$. We then have $\widetilde{\mathcal{V}}_i^\delta=\mathcal{V}\cap H_0^1(\widetilde{\Omega}_i^\delta)$. We define $\widetilde{\mathcal{V}}^\delta\subset\mathcal{V}^\delta$ as

$$\widetilde{\mathcal{V}}^\delta=\widetilde{\mathcal{V}}_1^\delta\oplus\cdots\oplus\widetilde{\mathcal{V}}_N^\delta,$$

which is a direct sum.

4.2 One- and two-level RASHO methods

Using notation introduced in the previous section, we now describe one-level RASHO method.

We first define two main operators :

- $\widetilde{P}_i^\delta : \widetilde{\mathcal{V}}^\delta \to \widetilde{\mathcal{V}}_i^\delta$ as a projection operator such that, for any $u \in \widetilde{\mathcal{V}}^\delta$

$$a(\widetilde{P}_i^\delta u, v) = a(u,v), \quad \forall v \in \widetilde{\mathcal{V}}_i^\delta.$$

- $\widetilde{R}_i^\delta$:

$$(\widetilde{R}_i^\delta v)(x_k) = \begin{cases} v_k, & x_k \in \widetilde{W}_i^\delta, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Where $v = (v_1, \cdots, v_n)^T$ is a vector corresponding to the nodal values of a function $u \in \mathcal{V}$, $x_k \in W$, $v_k = u(x_k)$. $\widetilde{R}_i^\delta$ is a restriction operator on $\widetilde{W}_i^\delta$.

Using this restriction operator, we define the subdomain stiffness matrix as

$$\widetilde{A}_i^\delta = \widetilde{R}_i^\delta A (\widetilde{R}_i^\delta)^T,$$

and matrix representation of $\widetilde{P}_i^\delta$ as

$$\widetilde{P}_i^\delta = (\widetilde{A}_i^\delta)^{-1} A.$$

Then, the one-level RASHO preconditioner is defined by

$$B = \sum_{i=1}^{N} (\widetilde{A}_i^\delta)^{-1}, \tag{4.2a}$$

$$BA = \sum_{i=1}^{N} \widetilde{P}_i^\delta. \tag{4.2b}$$

Following [4], if we use a coarse space $\widetilde{\mathcal{V}}_0^\delta$, then a two-level RASHO preconditioner can now be obtained bellow

$$B = \sum_{i=0}^{N} (\widetilde{A}_i^\delta)^{-1}, \tag{4.3a}$$

$$BA = \sum_{i=0}^{N} \widetilde{P}_i^\delta. \tag{4.3b}$$

5 Eigenvalue analysis of BA

In this section, we will estimate the condition number and effective condition number of BA in RASHO algorithm. And we will give the final convergence rate estimate of RASHO-PCG method.

Similar to $\widetilde{H}_D^1(\Omega)$ in Section 3, we introduce a subspace $\widetilde{\mathcal{V}}$ of $\widetilde{\mathcal{V}}^\delta$ by

$$\widetilde{\mathcal{V}} := \widetilde{H}_D^1(\Omega) \cap \widetilde{\mathcal{V}}^\delta = \Big\{ v \in \widetilde{\mathcal{V}}^\delta : \int_{\Omega_m} v = 0 \text{ for } m \in I \Big\}.$$

Then we have the following results:

Lemma 5.1. *For any $v\in\widetilde{\mathcal{V}}^\delta$, there exist $v_i\in\widetilde{\mathcal{V}}_i^\delta$ such that $v=\sum_{i=0}^N v_i$ and*

$$\sum_{i=0}^N a(v_i,v_i)\lesssim c_d(h,H)^2r^2a(v,v). \tag{5.1}$$

For any $v\in\widetilde{\mathcal{V}}$, there exist $v_i\in\widetilde{\mathcal{V}}_i^\delta$ such that $v=\sum_{i=0}^N v_i$ and

$$\sum_{i=0}^N a(v_i,v_i)\lesssim |\log H|r^2a(v,v), \tag{5.2}$$

where

$$r=\frac{H}{(2\delta+1)h}.$$

Proof. Here, we follow the ideas from [4]. Let $\{\theta_i\}_{i=1}^N$ be a partition of unity defined on Ω, consider $v=\sum_{i=0}^N v_i$ (taking $v_0=Q_0^\omega v$), $v\in\widetilde{\mathcal{V}}^\delta$, and $v_i\in\widetilde{\mathcal{V}}_i^\delta$ defined as follows :

$$v_i=I_h(\theta_i(v-Q_0^\omega v))\in\widetilde{\mathcal{V}}_i^\delta,\quad i=1,\cdots,N,$$

where I_h is the nodal value interpolant on $v\in\widetilde{\mathcal{V}}$.

Let $w=v-Q_0^\omega v$, for any $\tau\in\mathcal{T}_h$ we have

$$\begin{aligned}|v_i|_{1,\tau}^2&\le 2|\overline{\theta_i}w|_{1,\tau}^2+2|I_h(\overline{\theta_i}-\theta_i)w|_{1,\tau}^2\\ &\lesssim|w|_{1,\tau}^2+|I_h(\overline{\theta_i}-\theta_i)w|_{1,\tau}^2\\ &\lesssim|w|_{1,\tau}^2+h^{-2}\|I_h(\overline{\theta_i}-\theta_i)w\|_{0,\tau}^2.\end{aligned}$$

It is easy to show that :

$$h^{-2}\|I_h(\overline{\theta_i}-\theta_i)w\|_{0,\tau}^2\lesssim\frac{1}{((2\delta+1)h)^2}\|w\|_{0,\tau}^2.$$

Consequently,

$$|v_i|_{1,\tau}^2\lesssim|w|_{1,\tau}^2+\frac{1}{((2\delta+1)h)^2}\|w\|_{0,\tau}^2.$$

Summing over all $\tau\in\mathcal{T}_h\cap\Omega_i^\delta$ with appropriate weights gives

$$|v_i|_{1,\omega,\Omega_i^\delta}^2\lesssim|w|_{1,\omega,\Omega_i^\delta}^2+\frac{1}{((2\delta+1)h)^2}\|w\|_{0,\omega,\Omega_i^\delta}^2,$$

and

$$\begin{aligned}\sum_{i=0}^N a(v_i,v_i)&\lesssim\sum_i^N|v_i|_{1,\omega,\Omega_i^\delta}^2\lesssim\sum_{i=0}^N\Big(|w|_{1,\omega,\Omega_i^\delta}^2+\frac{1}{((2\delta+1)h)^2}\|w\|_{0,\omega,\Omega_i^\delta}^2\Big)\\ &\lesssim|v-Q_0^\omega v|_{1,\omega}^2+\frac{1}{((2\delta+1)h)^2}\|v-Q_0^\omega v\|_{0,\omega}^2.\end{aligned}$$

Applying Lemma 3.1 and let $r=H/(2\delta+1)h$, we obtain inequality (5.1). Similarly applying Lemma 3.2, we get inequality (5.2). $\square$

Lemma 5.2. *For the RASHO preconditioner B defined by* (4.3a), *the eigenvalues of BA satisfies*

$$\lambda_{\min}(BA) \geq c_d(h,H)^{-2}r^{-2},$$
$$\lambda_{m_0+1}(BA) \geq |\log H|^{-1}r^{-2},$$
$$\lambda_{\max}(BA) \leq C,$$

where

$$r = \frac{H}{(2\delta+1)h}.$$

Proof. For any $v=\sum_{i=0}^N v_i$, $v\in\widetilde{\mathcal{V}}^\delta$, and $v_i\in\widetilde{\mathcal{V}}_i^\delta$, using the Schwarz inequality, we obtain

$$\begin{aligned} a(v,v) &= \sum_{i=0}^N a(v_i,v) = \sum_{i=0}^N a(v_i,\widetilde{P}_i^\delta v) \\ &\leq \Big(\sum_{i=0}^N a(v_i,v_i)\Big)^{1/2}\Big(\sum_{i=0}^N a(\widetilde{P}_i^\delta v,\widetilde{P}_i^\delta v)\Big)^{1/2} \\ &= \Big(\sum_{i=0}^N a(v_i,v_i)\Big)^{1/2}(a(BAv,v))^{1/2}. \end{aligned}$$

Applying inequality (5.1), we get

$$a(v,v) \leq c_d(h,H)ra(v,v)^{1/2}a(BAv,v)^{1/2}.$$

This implies

$$\lambda_{\min}(BA) \geq c_d(h,H)^{-2}r^{-2}.$$

Applying inequality (5.2), we get

$$a(v,v) \leq |\log H|^{1/2}ra(v,v)^{1/2}a(BAv,v)^{1/2}.$$

And by (2.7), we get

$$\lambda_{m_0+1}(BA) \geq |\log H|^{-1}r^{-2}.$$

From [4], we also obtain

$$\lambda_{\max}(BA) \leq C.$$

Thus, we complete the proof. □

From Lemma 5.2, we observe that the condition number and effective condition number in RASHO algorithm can be controlled by $C_1c_d(h,H)^2r^2$ and $C_2|\log H|r^2$ respectively. Where C_1 and C_2 are constants which are independent with jump coefficients and meshsize. Then we can give convergence results of RASHO-PCG method as following.

Theorem 5.1. *In RASHO-PCG algorithm, we have the convergence rate estimate as*

$$\frac{\|u-u_k\|_A}{\|u-u_0\|_A} \leq 2(C_1 c_d(h,H)^2 r^2 - 1)^{m_0} \Big(1 - \frac{2}{C_2\sqrt{|\log H|}\cdot r+1}\Big)^{k-m_0} \quad \textit{for } k \geq m_0. \tag{5.3}$$

The number of iterations needed of

$$\frac{\|u-u_k\|_A}{\|u-u_0\|_A} < \epsilon$$

with the given tolerance $0<\varepsilon<1$ *satisfies*

$$k \geq m_0 + \frac{\log(\frac{2}{\epsilon}) + m_0 \log(C_1 c_d(h,H)^2 r^2 - 1)}{\log(\frac{C_2\sqrt{|\log H|}r+1}{C_2\sqrt{|\log H|}r-1})}, \tag{5.4}$$

where

$$r = \frac{H}{(2\delta+1)h}.$$

6 Conclusions

In this paper, we provided a proof of robustness of the restricted additive Schwarz preconditioners with harmonic overlap (RASHO) for the second order elliptic problems with jump coefficients. We discussed the eigenvalue distribution of the RASHO-preconditioner and found that only a few small eigenvalues infected by the large jump, and effective condition number can be bounded by $\mathcal{O}(|\log H| r^2)$, which is uniform bounded with respect to the coefficients and meshsize and overlapping factor δ. Finally, we given the convergence rate estimate of restricted additive Schwarz method.

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