Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **8**, No. 6, pp. 1072-1083

DOI: 10.4208/aamm.2014.m669
December 2016

Restricted Additive Schwarz Preconditioner for Elliptic Equations with Jump Coefficients

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Received 21 June 2014; Accepted (in revised version) 23 November 2015

Abstract. This paper provides a proof of robustness of the restricted additive Schwarz preconditioner with harmonic overlap (RASHO) for the second order elliptic problems with jump coefficients. By analyzing the eigenvalue distribution of the RASHO preconditioner, we prove that the convergence rate of preconditioned conjugate gradient method with RASHO preconditioner is uniform with respect to the large jump and meshsize.

AMS subject classifications: 65N30, 65N55, 65F10

Key words: Jump coefficients, conjugate gradient, effective condition number, domain decomposition, restricted additive Schwarz.

1 Introduction

In this paper, we will discuss the restricted additive Schwarz (with harmonic overlap) preconditioned conjugate gradient method for the linear finite element approximation of the second order elliptic boundary value problem

$$\begin{cases} -\nabla\cdot(\omega\nabla u)=f & \text{in } \Omega, \\ u=g_D & \text{on } \Gamma_D, \\ -\omega\dfrac{\partial u}{\partial n}=g_N & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\Omega\in R^d$ $(d=1,2 \text{ or } 3)$ is a polygonal or polyhedral domain with Dirichlet boundary Γ_D and Neumann boundary Γ_N. The coefficient $\omega=\omega(x)$ is a positive and piecewise

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constant function. More precisely, we assume that there are M open disjointed polygonal or polyhedral regions Ω_m $(m=1,\cdots,M)$ satisfying $\cup_{m=1}^{M}\overline{\Omega}_m=\overline{\Omega}$ with

$$\omega_m=\omega|_{\Omega_m},\quad m=1,\cdots,M,$$

where each $\omega_m>0$ is a constant. The analysis can be carried through to a more general case when $\omega(x)$ varies moderately in each subdomain.

The goal of this paper is to prove the robustness of restricted additive Schwarz preconditioner with harmonic overlap (RASHO-PCG). The design of RASHO method is based on a much deeper understanding of the behavior of Schwarz-type methods (see [4] and the references cited therein). It is a modification and symmetrized version of RAS (introduced in [3]). In this paper, we solve numerically the second elliptic problem with jump coefficients by RASHO-PCG method. As a result, we prove that the convergence rate of RASHO-PCG method is uniform with respect to the large jump and meshsize. We will see that the effective condition number of RASHO-PCG is $C|\log H|\cdot r^2$.

The rest of the paper is organized as follows. In order to make the paper be self-contained, we refer directly to parts of contents in [9] and [10]. In Section 2, we introduce some basic notation, the PCG algorithm and some theoretical foundations. In Section 3, we quote some main results on the weighted L^2-projection from [2]. Section 4 is an introduction of the RASHO preconditioner. In Section 5, we analyze the eigenvalue distribution of the RASHO preconditioned system and prove the convergence rate of the PCG algorithm. In Section 6, we give some conclusion remarks.

2 Preliminaries

2.1 Notation

We introduce the bilinear form

$$a(u,v)=\sum_{m=1}^{M}\omega_m(\nabla u,\nabla v)_{L^2(\Omega_m)},\quad \forall u,v\in H_D^1(\Omega),$$

where $H_D^1(\Omega)=\{v\in H^1(\Omega):v|_{\Gamma_D}=0\}$, and introduce the H^1-norm and seminorm with respect to any subregion Ω_m by

$$|u|_{1,\Omega_m}=\|\nabla u\|_{0,\Omega_m},\quad \|u\|_{1,\Omega_m}=(\|u\|_{0,\Omega_m}^2+|u|_{1,\Omega_m}^2)^{\frac{1}{2}}.$$

Thus,

$$a(u,u)=\sum_{m=1}^{M}\omega_m|u|_{1,\Omega_m}^2:=|u|_{1,\omega}^2.$$

We also need the weighted L^2-inner product

$$(u,v)_{0,\omega}=\sum_{m=1}^{M}\omega_m(u,v)_{L^2(\Omega_m)}$$