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Two-Level Defect-Correction Method for Steady Navier-Stokes Problem with Friction Boundary Conditions

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Abstract. In this paper, we present two-level defect-correction finite element method for steady Navier-Stokes equations at high Reynolds number with the friction boundary conditions, which results in a variational inequality problem of the second kind. Based on Taylor-Hood element, we solve a variational inequality problem of Navier-Stokes type on the coarse mesh and solve a variational inequality problem of Navier-Stokes type corresponding to Newton linearization on the fine mesh. The error estimates for the velocity in the $\mathbf{H}^1$ norm and the pressure in the L^2 norm are derived. Finally, the numerical results are provided to confirm our theoretical analysis.

AMS subject classifications: 65N30, 76M10

Key words: Navier-Stokes equations, friction boundary conditions, variational inequality problems, defect-correction method, two-level mesh method.

1 Introduction

Let $\Omega\subset\mathbb{R}^2$ be a bounded and convex domain with Lipschitz boundary $\partial\Omega$. Consider steady incompressible flows which are governed by:

$$\begin{cases} -\mu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla p=\mathbf{f} & \text{in } \Omega, \\ \text{div }\mathbf{u}=0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u}=(u_1,u_2)$ denotes the velocity vector of the flows, p the pressure and $\mathbf{f}=(f_1,f_2)$ the body force vector. The constant $\mu=1/Re>0$ is the viscosity with Reynolds number Re. In this paper, the following friction boundary conditions are considered:

$$\begin{cases} \mathbf{u}=0 & \text{on } \Gamma, \\ \mathbf{u}_\mathbf{n}=\mathbf{u}\cdot\mathbf{n}=0, \quad -\sigma_\tau(\mathbf{u})\in g\partial|\mathbf{u}_\tau| & \text{on } S, \end{cases} \tag{1.2}$$

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where $\Gamma\cap S=\emptyset$ and $\overline{\Gamma\cup S}=\partial\Omega$. g is a scalar function. $\mathbf{n}$ represents the unit vector of the external normal to S. $\mathbf{u}_\tau$ and $\sigma_\tau(\mathbf{u})$ are the tangential components of the velocity and the stress vector σ which is defined by $\sigma_i=\sigma_i(\mathbf{u},p)=(\mu e_{ij}(\mathbf{u})-p\delta_{ij})n_j$ with $e_{ij}(\mathbf{u})=\frac{\partial u_i}{\partial x^j}+\frac{\partial u_j}{\partial x^i}$, $i,j=1,2$. The subdifferential set is defined as follows. Let ψ be a given function which is of convexity and weak semi-continuity from below. The subdifferentialset $\partial\psi(a)$ is defined by

$$\partial\psi(a)=\{b\in\mathbb{R}:\psi(h)-\psi(a)\geq b(h-a),\ \ \forall h\in\mathbb{R}\}.$$

The boundary conditions (1.2) were introduced by H. Fujita to describe some problems in hydrodynamics [5]. Some well-posedness results from the view of theory have been studied, such as R. An, Y. Li and K. Li [1], H. Fujita [6–8], T. Kashiwabara [15], Y. Li and K. Li [26, 28], Le Roux [31, 32], N. Saito [33] and references cited therein. Although there are a large amount of works about the finite element methods for Navier-Stokes equations, however, the numerical methods for the problem (1.1)-(1.2) have not been studied as much. The reason is that the variational formulation of (1.1)-(1.2) is of the form of variational inequality due to the subdifferential property on the boundary S. M. Ayadi, M. Gdoura and T. Sassi studied mini-element method for Stokes problem in [3]. T.Kashiwabara studied optimal finite element error bounds by defining the different numerical integration of the non-differential term on the boundary S corresponding to the different finite element pairs [16, 17]. The penalty and stabilized finite element methods and their two-level mesh methods for steady problem were studied in [2, 4, 22–25]. In these works, all numerical experiments were displayed only for small Reynolds number. It is well known that for the incompressible flows at high Reynolds number, Navier-Stokes equations are the domination of the convection and the flows are very unstable. Thus, it is difficult to make the numerical simulation of the incompressible flows efficiently.

There are some stabilized methods to overcome the difficulty in simulating the incompressible flows at high Reynolds number numerically, such as the variational multiscale method [13, 14, 34, 35], the subgrid method [11, 21], the defect-correction method [18–20, 29], etc. The defect-correction method is an iterative improvement technique and can increase the accuracy of the solution without refining the mesh, so it has been successively applied to Navier-Stokes equations at high Reynolds number. W. Layton firstly studied defect-correction method for the steady incompressible flows at high Reynolds number in [19]. Recently, H. Qiu and L. Mei studied the two-level defect-correction method for steady Navier-Stokes problem by using the stabilized finite element method [30].

In this paper, we combine defect-correction method with two-level mesh technique to solve the problem (1.1) at high Reynolds number with friction boundary conditions (1.2) numerically. Since the variational formulation of the problem (1.1)-(1.2) is the variational inequality problem, there exist some differences between our method for the problem (1.1)-(1.2) and those for Navier-Stokes equations (1.1) with Dirichlet boundary conditions. The main idea of our two-level method is to solve a nonlinear variational inequality problem of Navier-Stokes type at the defect step on the coarse mesh and solve a linearized variational inequality problem at the correction step on the fine mesh. Moreover, we have to construct appropriate iterative algorithms to solve two variational inequality subproblems in practical computations.

2 Preliminary

Let $L^2(\Omega)$ denote the Lebesgue space of square-integrable functions over Ω with norm $\|\cdot\|$ and inner-product $(\cdot,\cdot)$. Let $H^m(\Omega)$ with $m\in\mathbb{N}$ denote the Sobolev space of all functions having square integrable derivatives up to order m over Ω with the classical Sobolev norm $\|\cdot\|_m$. We use the boldface type $\mathbf{H}^m(\Omega)$ and $\mathbf{L}^2(\Omega)$ to denote the vector Sobolev spaces $H^m(\Omega)^2$ and $L^2(\Omega)^2$, respectively. Throughout this paper, the symbols $C,C_0,C_1,\cdots$ are used to denote some positive constant which are independent of the mesh parameter h,H, the viscosity μ, and may take different values even in the same formulation.

For the mathematical setting, we introduce the following function spaces:

$$\mathbf{V}=\{\mathbf{u}\in\mathbf{H}^1(\Omega),\ \mathbf{u}|_\Gamma=0,\ \mathbf{u}\cdot\mathbf{n}|_S=0\},\qquad \mathbf{V}_0=\mathbf{H}_0^1(\Omega),$$
$$\mathbf{V}_\sigma=\{\mathbf{u}\in\mathbf{V},\ \text{div}\ \mathbf{u}=0\},\qquad M=L_0^2(\Omega)=\Big\{q\in L^2(\Omega),\ \int_\Omega q dx=0\Big\}.$$

The norm in $\mathbf{V}$ is defined by

$$\|\mathbf{v}\|_V=\Big(\int_\Omega|\nabla\mathbf{v}|^2dx\Big)^{1/2},\quad \forall\mathbf{v}\in\mathbf{V}.$$

Then $\|\cdot\|_V$ is equivalent to $\|\cdot\|_1$ due to Poincaré inequality. We introduce the following bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$ and $\mathbf{V}\times M$, respectively, by

$$a(\mathbf{u},\mathbf{v})=\mu\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}dx,\quad d(\mathbf{v},q)=\int_\Omega q\text{div}\ \mathbf{v}dx,$$

and a trilinear form on $\mathbf{V}\times\mathbf{V}\times\mathbf{V}$ by

$$\begin{aligned}b(\mathbf{u},\mathbf{v},\mathbf{w})&=\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}dx+\frac{1}{2}\int_\Omega(\text{div}\ \mathbf{u})\mathbf{v}\cdot\mathbf{w}dx\\&=\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}dx-\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}dx.\end{aligned}$$

For all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}$, it is easy to check that $b(\mathbf{u},\mathbf{v},\mathbf{w})$ satisfies:

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v}),\tag{2.1a}$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w})\le N\|\mathbf{u}\|_V\|\mathbf{v}\|_V\|\mathbf{w}\|_V.\tag{2.1b}$$

Given $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $g\in L^2(S)$ with $g>0$ on S, based on the above notations, the variational formulation of the problem (1.1)-(1.2) reads as: find $(\mathbf{u},p)\in\mathbf{V}\times M$ such that for all $(\mathbf{v},q)\in\mathbf{V}\times M$

$$\begin{cases} a(\mathbf{u},\mathbf{v}-\mathbf{u})+b(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u})+j(\mathbf{v}_\tau)-j(\mathbf{u}_\tau)-d(\mathbf{v}-\mathbf{u},p)\geq(\mathbf{f},\mathbf{v}-\mathbf{u}), \\ d(\mathbf{u},q)=0, \end{cases} \tag{2.2}$$

with $j(\mathbf{v}_\tau)=\int_S g|\mathbf{v}_\tau|ds$. It is obvious that the problem (2.2) is the variational inequality problem of the second kind with Navier-Stokes operator. N. Saito in [33] has proved that $b(\cdot,\cdot)$ satisfies the inf-sup condition on $\mathbf{V}\times M$, i.e., there exists some positive $\beta_0>0$ such that

$$\beta_0\|q\|\leq\sup_{\mathbf{v}\in\mathbf{V}}\frac{d(\mathbf{v},q)}{\|\mathbf{v}\|_V}.$$

Then the variational inequality problem (2.2) is equivalent to: find $\mathbf{u}\in\mathbf{V}_\sigma$ such that

$$a(\mathbf{u},\mathbf{v}-\mathbf{u})+b(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u})+j(\mathbf{v}_\tau)-j(\mathbf{u}_\tau)\geq(\mathbf{f},\mathbf{v}-\mathbf{u}),\quad\forall\mathbf{v}\in\mathbf{V}_\sigma. \tag{2.3}$$

Now we recall the existence and uniqueness result of the solution to the problem (2.3) established in [25].

Theorem 2.1. *If the following uniqueness condition holds:*

$$\frac{4\kappa_1N(\|\boldsymbol{f}\|+\|g\|_{L^2(S)})}{\mu^2}<1, \tag{2.4}$$

then the variational inequality problem (2.3) *admits a unique solution* $\boldsymbol{u}\in\boldsymbol{V}_\sigma$ *satisfying*

$$\|\boldsymbol{u}\|_V\leq\frac{\kappa_1}{\mu}(\|\boldsymbol{f}\|+\|g\|_{L^2(S)}), \tag{2.5}$$

where $\kappa_1>0$ *satisfies*

$$|(\boldsymbol{f},\boldsymbol{v})-j(\boldsymbol{v}_\tau)|\leq\kappa_1(\|\boldsymbol{f}\|+\|g\|_{L^2(S)})\|\boldsymbol{v}\|_V,\quad\forall\boldsymbol{v}\in\boldsymbol{V}_\sigma.$$

Remark 2.1. Since the problem (2.2) is of the form of variational inequality, then the appropriate iteration algorithms are needed in the practical computation. In this paper, we use Uzawa iteration algorithms in [16, 27], which are based on the following equivalent variational equation of the problem (2.2): find $(\mathbf{u},p,\lambda)\in\mathbf{V}\times M\times\Lambda$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})-d(\mathbf{v},p)+\int_S g\lambda\mathbf{v}_\tau ds=(\mathbf{f},\mathbf{v}), & \forall\mathbf{v}\in\mathbf{V}, \\ d(\mathbf{u},q)=0, & \forall q\in M, \\ \lambda\mathbf{u}_\tau=|\mathbf{u}_\tau| & \text{on } S, \end{cases} \tag{2.6}$$

where

$$\Lambda=\{\gamma\in L^2(S):\ |\gamma(x)|\leq1\text{ on }S\}.$$

The equivalence can be proved by using a similar way for Theorem 3.1 in [27].

3 Two-level defect-correction method

As mentioned in Section 1, the classical Galerkin finite element methods become unstable for the numerical approximation of the incompressible flows at high Reynolds number. In this section, we give the defect-correction finite element approximation. Let $\mathcal{T}_\eta=\{K\}$ be a quasi-uniform family of triangular partition of Ω into triangles of diameter not greater than $0<\eta<1$. The corresponding ordered triangles are denoted by $K_1,K_2,\cdots,K_M$. Let $\eta_i=\text{diam}(K_i)$, $i=1,\cdots,M$, and $\eta=\max\{\eta_1,\eta_2,\cdots,\eta_M\}$. Let $e'=\partial K\cap S$. $\{Q_i\}_{i=1}^{m+1}$ are all vertices of triangles in $\mathcal{T}_\eta$, which are located in $\overline{S}$ and arranged in ascending order along $\overline{S}$. Then $\overline{S}\cap\overline{\Gamma}=\{Q_1,Q_{m+1}\}$. Let $P_r(K)$ be the space of the polynomials on $K\in\mathcal{T}_\eta$ of degree at most r. Based on Taylor-Hood element, define the finite element subspaces of $\mathbf{V}$ and M, respectively, by

$$\mathbf{W}_\eta=\{\mathbf{v}_\eta\in\mathbf{C}(\overline{\Omega}),\ \mathbf{v}_\eta|_K\in\mathbf{P}_2(K),\ \forall K\in\mathcal{T}_\eta\},\quad \mathbf{V}_\eta=\mathbf{W}_\eta\cap\mathbf{V},$$
$$\mathbf{V}_{0\eta}=\mathbf{W}_\eta\cap\mathbf{V}_0\subset\mathbf{V}_\eta,\quad M_\eta=\{q_\eta\in C(\overline{\Omega}),\ q_\eta|_K\in P_1(K),\ \forall K\in\mathcal{T}_\eta\}\cap M.$$

For approximating functions defined on the boundary S, we define

$$\Lambda_\eta=\{\mu_\eta\in C(\overline{S}),\ \mu_\eta|_{e'}\in P_2(e'),\ \mu_\eta(Q_1)=\mu_\eta(Q_{m+1})=0\}\cap\Lambda.$$

For each $\mu_\eta\in\Lambda_\eta$, the discrete projection operator P_{Λ_η} is defined by

$$P_{\Lambda_\eta}(\mu_\eta)=\begin{cases}+1, & \text{if } \mu_\eta(Q)>1,\\ \mu_\eta(Q), & \text{if } |\mu_\eta(Q)|\le 1,\\ -1, & \text{if } \mu_\eta(Q)<-1,\end{cases}\qquad \forall Q\in\{Q_i\}_{i=1}^{m+1}.$$

Under the above choice of $\mathbf{V}_\eta$ and M_η, there exists some positive constant $\beta_1>0$ independent of η such that

$$\beta_1\|q_\eta\|\le\sup_{\mathbf{v}_\eta\in\mathbf{V}_{0\eta}}\frac{d(\mathbf{v}_\eta,q_\eta)}{\|\mathbf{v}_\eta\|_V}. \tag{3.1}$$

To obtain the error estimates, we assume the following approximation properties:

$$\inf_{\mathbf{v}_\eta\in\mathbf{V}_\eta}\{\|\mathbf{v}-\mathbf{v}_\eta\|+\eta\|\mathbf{v}-\mathbf{v}_\eta\|_V\}\le C\eta^3\|\mathbf{v}\|_3,\qquad \forall\mathbf{v}\in\mathbf{H}^3(\Omega), \tag{3.2a}$$
$$\inf_{q_\eta\in M_\eta}\|q-q_\eta\|\le C\eta^2\|q\|_2,\qquad \forall q\in H^2(\Omega). \tag{3.2b}$$

From the trace inequality $\|\mathbf{v}\|_{L^2(S)}\le C\|\mathbf{v}\|^{1/2}\|\mathbf{v}\|_V^{1/2}$, one has

$$\inf_{\mathbf{v}_\eta\in\mathbf{V}_\eta}\|\mathbf{v}-\mathbf{v}_\eta\|_{L^2(S)}\le C\eta^{5/2}\|\mathbf{v}\|_3,\quad \forall\mathbf{v}\in\mathbf{H}^3(\Omega). \tag{3.3}$$

From now on, H and h with $h<H$ are two real positive parameter. The fine mesh partition $\mathcal{T}_h$ is generated by a mesh refinement process from the coarse mesh partition $\mathcal{T}_H$. The finite element space pairs $(\mathbf{V}_h,M_h)$ and $(\mathbf{V}_H,M_H)$ are corresponding to the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. Let λ_1 and λ_2 be two stabilized parameters. The two-level defect-correction finite element method for the approximation of the problem (2.2) is constructed as follows:

Step I: Solve a defect Navier-Stokes type variational inequality problem on the coarse mesh:

$$\begin{cases} \text{find } (\mathbf{u}_H,p_H)\in\mathbf{V}_H\times M_H \text{ such that} \\ a(\mathbf{u}_H,\mathbf{v}_H-\mathbf{u}_H)-d(\mathbf{v}_H-\mathbf{u}_H,p_H)+b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{v}_H-\mathbf{u}_H) \\ \qquad +\dfrac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{v}_H-\mathbf{u}_H)+j(\mathbf{v}_{H\tau})-j(\mathbf{u}_{H\tau})\ge(\mathbf{f},\mathbf{v}_H-\mathbf{u}_H), \quad \forall\mathbf{v}_H\in\mathbf{V}_H, \\ d(\mathbf{u}_H,q_H)=0, \quad \forall q_H\in M_H. \end{cases} \tag{3.4}$$

Step II: Solve a correction Navier-Stokes type variational inequality problem corresponding to Newton linearization on the fine mesh:

$$\begin{cases} \text{find } (\mathbf{u}_h,p_h)\in\mathbf{V}_h\times M_h \text{ such that} \\ a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)-d(\mathbf{v}_h-\mathbf{u}_h,p_h)+b(\mathbf{u}_H,\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)+b(\mathbf{u}_h,\mathbf{u}_H,\mathbf{v}_h-\mathbf{u}_h) \\ \qquad +\dfrac{\lambda_2 h}{\mu}a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)+j(\mathbf{v}_{h\tau})-j(\mathbf{u}_{h\tau}) \\ \qquad \ge(\mathbf{f},\mathbf{v}_h-\mathbf{u}_h)+b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{v}_h-\mathbf{u}_h)+\dfrac{\lambda_2 h}{\mu}a(\mathbf{u}_H,\mathbf{v}_h-\mathbf{u}_h), \quad \forall\mathbf{v}_h\in\mathbf{V}_h, \\ d(\mathbf{u}_h,q_h)=0, \qquad \forall q_h\in M_h. \end{cases} \tag{3.5}$$

First, we estimate the solution $\mathbf{u}_H$ by the classical argument on the coarse mesh under the uniqueness condition (2.4). Taking $(\mathbf{v}_H,q_H)=(0,p_H)$ and $(\mathbf{v}_H,q_H)=(2\mathbf{u}_H,p_H)$ in (3.4), we get

$$a(\mathbf{u}_H,\mathbf{u}_H)+\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{u}_H)=(f,\mathbf{u}_H)-j(\mathbf{u}_{H\tau}),$$

from which one has

$$\|\mathbf{u}_H\|_V\le\frac{\kappa_1}{\mu+\lambda_1 H}(\|\mathbf{f}\|+\|g\|_{L^2(S)})<\frac{\mu}{4N}. \tag{3.6}$$

The existence and uniqueness of $(\mathbf{u}_H,p_H)\in\mathbf{V}_H\times M_H$ follows from the classical results of the variational inequality problem (see Theorem 1.6.1 in [10]) and Navier-Stokes equations (see Theorem 4.3.1 in [9]) in finite element spaces. Moreover, we can prove that the finite element approximation solution $(\mathbf{u}_H,p_H)$ is of the following error estimate.

Theorem 3.1. *Under the uniqueness condition* (2.4)*, let* $(\boldsymbol{u},p)\in\boldsymbol{V}\times M$ *and* $(\boldsymbol{u}_H,p_H)\in\boldsymbol{V}_H\times M_H$ *be the solutions of* (2.2) *and* (3.4)*, respectively. Then there exists some* $C_0>0$ *such that*

$$\begin{aligned}&\mu\|\boldsymbol{u}-\boldsymbol{u}_H\|_V+\|p-p_H\|\\ \le& C_0\Big(\inf_{\boldsymbol{v}_H\in\boldsymbol{V}_H}\mu\|\boldsymbol{u}-\boldsymbol{v}_H\|_V+\inf_{\boldsymbol{v}_H\in\boldsymbol{V}_H}\sqrt{\mu}\|\boldsymbol{u}_\tau-\boldsymbol{v}_{H\tau}\|^{1/2}_{L^2(S)}+\inf_{q_H\in M_H}\|p-q_H\|+\mu\lambda_1 H\Big).\end{aligned}\tag{3.7}$$

Proof. Taking $\mathbf{v}=\mathbf{u}_H$ and $\mathbf{v}=2\mathbf{u}-\mathbf{v}_H$ in the first inequality of (2.2) and summing up the resulting inequalities yield

$$\begin{aligned}&a(\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)+b(\mathbf{u},\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)+j(\mathbf{u}_{H\tau})-2j(\mathbf{u}_\tau)\\&+j(2\mathbf{u}_\tau-\mathbf{v}_{H\tau})-d(\mathbf{u}_H-\mathbf{v}_H,p)\ge(f,\mathbf{u}_H-\mathbf{v}_H).\end{aligned}$$

From (3.4), one has

$$\begin{aligned}&a(\mathbf{u}_H-\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)\\ \le& b(\mathbf{u},\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)-b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{u}_H-\mathbf{v}_H)+j(\mathbf{v}_{H\tau})-2j(\mathbf{u}_\tau)\\&+j(2\mathbf{u}_\tau-\mathbf{v}_{H\tau})-d(\mathbf{u}_H-\mathbf{v}_H,p-p_H)-\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{u}_H-\mathbf{v}_H).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mu\|\mathbf{u}_H-\mathbf{v}_H\|_V^2\\ \le&|a(\mathbf{u}-\mathbf{v}_H,\mathbf{u}_H-\mathbf{v}_H)|+|b(\mathbf{u},\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)-b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{u}_H-\mathbf{v}_H)|\\&+|j(\mathbf{v}_{H\tau})-2j(\mathbf{u}_\tau)+j(2\mathbf{u}_\tau-\mathbf{v}_{H\tau})|+|d(\mathbf{u}_H-\mathbf{v}_H,p-p_H)|\\&+\Big|\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{u}_H-\mathbf{v}_H)\Big|\\ =&I_1+\cdots+I_5.\end{aligned}\tag{3.8}$$

From Hölder inequality and Young inequality, I_1 and I_5 can be estimated, respectively, by

$$I_1\le\mu\|\mathbf{u}-\mathbf{v}_H\|_V\|\mathbf{u}_H-\mathbf{v}_H\|_V\le\frac{\mu}{8}\|\mathbf{u}_H-\mathbf{v}_H\|_V^2+2\mu\|\mathbf{u}-\mathbf{v}_H\|_V^2,$$

and

$$\begin{aligned}I_5\le&\lambda_1 H\|\mathbf{u}_H\|_V\|\mathbf{u}_H-\mathbf{v}_H\|_V\le\frac{\mu\lambda_1 H}{4N}\|\mathbf{u}_H-\mathbf{v}_H\|_V\\ \le&\frac{\mu}{8}\|\mathbf{u}_H-\mathbf{v}_H\|_V^2+\frac{\mu\lambda_1^2H^2}{8N^2}.\end{aligned}$$

We rewrite I_2 as

$$\begin{aligned}&b(\mathbf{u},\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)-b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{u}_H-\mathbf{v}_H)\\ =&b(\mathbf{u}-\mathbf{v}_H,\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)+b(\mathbf{v}_H-\mathbf{u}_H,\mathbf{u},\mathbf{u}_H-\mathbf{v}_H)+b(\mathbf{u}_H,\mathbf{u}-\mathbf{v}_H,\mathbf{u}_H-\mathbf{v}_H).\end{aligned}$$

Then from (2.1b), (2.4), (2.5) and (3.6), I_2 is bounded by

$$\begin{aligned}I_2 \le & N(\|\mathbf{u}\|_V|+\|\mathbf{u}_H\|_V)\|\mathbf{u}-\mathbf{v}_H\|_V\|\mathbf{u}_H-\mathbf{v}_H\|_V+N\|\mathbf{u}\|_V\|\mathbf{u}_H-\mathbf{v}_H\|_V^2\\ \le & \frac{\mu}{2}\|\mathbf{u}-\mathbf{v}_H\|_V\|\mathbf{u}_H-\mathbf{v}_H\|_V+\frac{\mu}{4}\|\mathbf{u}_H-\mathbf{v}_H\|_V^2\\ \le & \frac{\mu}{2}\|\mathbf{u}_H-\mathbf{v}_H\|_V^2+\frac{\mu}{4}\|\mathbf{u}-\mathbf{v}_H\|_V^2.\end{aligned}$$

Using the triangular inequality, we estimate I_3 as

$$I_3 \le 2\|g\|_{L^2(S)}\|\mathbf{u}_\tau-\mathbf{v}_{H\tau}\|_{L^2(S)}.$$

By the following identity

$$d(\mathbf{u}_H-\mathbf{v}_H,p-p_H)=d(\mathbf{u}_H-\mathbf{v}_H,p-q_H)+d(\mathbf{u}-\mathbf{v}_H,q_H-p_H),$$

I_4 satisfies

$$\begin{aligned}I_4 \le & \|\mathbf{u}_H-\mathbf{v}_H\|_V\|p-q_H\|+\|\mathbf{u}-\mathbf{v}_H\|_V\|q_H-p_H\|\\ \le & \frac{\mu}{8}\|\mathbf{u}_H-\mathbf{v}_H\|_V^2+\frac{2}{\mu}\|p-q_H\|^2+\varepsilon_1^2\|q_H-p_H\|^2+\frac{1}{4\varepsilon_1^2}\|\mathbf{u}-\mathbf{v}_H\|_V^2,\end{aligned}$$

where ε_1 is some positive constant determined later. Combining the above estimates into (3.8), we get

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_H\|_V \le & C_1\Big(\|\mathbf{u}-\mathbf{v}_H\|_V+\frac{1}{\sqrt{\mu}}\|\mathbf{u}_\tau-\mathbf{v}_{H\tau}\|_{L^2(S)}^{1/2}+\frac{1}{\mu}\|p-q_H\|+\lambda_1 H\Big)\\ & +\frac{C_2\varepsilon_1}{\sqrt{\mu}}\|q_H-p_H\|+\frac{1}{2\varepsilon_1\sqrt{\mu}}\|\mathbf{u}-\mathbf{v}_H\|.\end{aligned} \tag{3.9}$$

Next, we estimate $\|p_H-q_H\|$ in terms of (3.1). For all $\mathbf{w}_H\in\mathbf{V}_{0H}$, taking $\mathbf{v}=\mathbf{u}\pm\mathbf{w}_H$ and $\mathbf{v}_H=\mathbf{u}_H\pm\mathbf{w}_H$ in the first inequalities of (2.2) and (3.4), and subtracting the resulting equations lead to

$$a(\mathbf{u}-\mathbf{u}_H,\mathbf{w}_H)+b(\mathbf{u},\mathbf{u},\mathbf{w}_H)-b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{w}_H)-\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{w}_H)=d(\mathbf{w}_H,p-p_H).$$

Then we have

$$\begin{aligned}& d(\mathbf{w}_H,q_H-p_H)\\ =& d(\mathbf{w}_H,q_H-p)+a(\mathbf{u}-\mathbf{u}_H,\mathbf{w}_H)+b(\mathbf{u},\mathbf{u},\mathbf{w}_H)-b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{w}_H)-\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{w}_H)\\ =& d(\mathbf{w}_H,q_H-p)+a(\mathbf{u}-\mathbf{u}_H,\mathbf{w}_H)+b(\mathbf{u}-\mathbf{u}_H,\mathbf{u},\mathbf{w}_H)+b(\mathbf{u}_H,\mathbf{u}-\mathbf{u}_H,\mathbf{w}_H)-\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{w}_H).\end{aligned}$$

It follows from (3.1) that

$$\begin{aligned}\beta_1\|q_H-p_H\| &\le \|p-q_H\|+(N(\|\mathbf{u}\|_V+\|\mathbf{u}_H\|_V)+\mu)\|\mathbf{u}-\mathbf{u}_H\|_V+\lambda_1 H\|u_H\|_V\\ &\le \|p-q_H\|+\frac{3\mu}{2}\|\mathbf{u}-\mathbf{u}_H\|_V+\frac{\mu\lambda_1 H}{4N}.\end{aligned} \tag{3.10}$$

Combining (3.10) into (3.9) and taking

$$\varepsilon_1=\frac{\beta_1}{6C_2\sqrt{\mu}},$$

we obtain

$$\mu\|\mathbf{u}-\mathbf{u}_H\|_V \le \frac{C_0}{2}\big(\mu\|\mathbf{u}-\mathbf{v}_H\|_V+\sqrt{\mu}\|\mathbf{u}_\tau-\mathbf{v}_{H\tau}\|^{1/2}_{L^2(S)}+\|p-q_H\|+\mu\lambda_1 H\big).$$

The error estimate (3.7) for the pressure also holds if we use (3.10) again. □

3.1 Oseen linearization on coarse mesh

In order to construct Uzawa algorithms described in Section 4, the Oseen linearization is used to linearize the nonlinear term in (3.4). Given the iteration initial value $(\mathbf{u}_H^0,p_H^0)$ which is defined by

$$\begin{cases} a(\mathbf{u}_H^0,\mathbf{v}_H-\mathbf{u}_H^0)-d(\mathbf{v}_H-\mathbf{u}_H^0,p_H^0)+j(\mathbf{v}_{H\tau})-j(\mathbf{u}_{H\tau}^0)\ge(\mathbf{f},\mathbf{v}_H-\mathbf{u}_H^0),\\ d(\mathbf{u}_H^0,q_H)=0,\end{cases} \tag{3.11}$$

for all $(\mathbf{v}_H,q_H)\in\mathbf{V}_H\times M_H$, we solve $(\mathbf{u}_H^n,p_H^n)\in\mathbf{V}_H\times M_H$, $n=1,2,\cdots$, by

$$\begin{cases} a(\mathbf{u}_H^n,\mathbf{v}_H-\mathbf{u}_H^n)-d(\mathbf{v}_H-\mathbf{u}_H^n,p_H^n)+b(\mathbf{u}_H^{n-1},\mathbf{u}_H^n,\mathbf{v}_H-\mathbf{u}_H^n)\\ \quad+\dfrac{\lambda_1 H}{\mu}a(\mathbf{u}_H^n,\mathbf{v}_H-\mathbf{u}_H^n)+j(\mathbf{v}_{H\tau})-j(\mathbf{u}_{H\tau}^n)\ge(\mathbf{f},\mathbf{v}_H-\mathbf{u}_H^n),\\ d(\mathbf{u}_H^n,q_H)=0.\end{cases} \tag{3.12}$$

By the classical argument, it is easily shown that $\mathbf{u}_H^0$ and $\mathbf{u}_H^n$ satisfy

$$\|\mathbf{u}_H^0\|_V\le\frac{\kappa_1}{\mu}(\|\mathbf{f}\|+\|g\|_{L^2(S)})<\frac{\mu}{4N}, \tag{3.13}$$

and

$$\|\mathbf{u}_H^n\|_V\le\frac{\kappa_1}{\mu+\lambda_1 H}(\|\mathbf{f}\|+\|g\|_{L^2(S)})<\frac{\mu}{4N}. \tag{3.14}$$

Now, we begin to prove the error estimates for $(\mathbf{u}_H^n,p_H^n)$. First, we have the following lemmas.

Lemma 3.1. *Under the uniqueness condition* (2.4), *let* $(\boldsymbol{u}_H,p_H)\in \boldsymbol{V}_H\times M_H$ *and* $(\boldsymbol{u}_H^0,p_H^0)\in \boldsymbol{V}_H\times M_H$ *be the solutions of* (3.4) *and* (3.11), *respectively. Then we have*

$$\mu\|\boldsymbol{u}_H-\boldsymbol{u}_H^0\|_V\le\frac{\kappa_1(\mu+4\lambda_1 H)}{4\mu+4\lambda_1 H}(\|\boldsymbol{f}\|+\|g\|_{L^2(S)}),\tag{3.15a}$$

$$\beta_1\|p_H-p_H^0\|\le\frac{2\kappa_1(\mu+4\lambda_1 H)}{4\mu+4\lambda_1 H}(\|\boldsymbol{f}\|+\|g\|_{L^2(S)}).\tag{3.15b}$$

Proof. Taking $(\mathbf{v}_H,q_H)=(\mathbf{u}_H^0,p_H^0-p_H)$ in (3.4) and $(\mathbf{v}_H,q_H)=(\mathbf{u}_H,p_H-p_H^0)$ in (3.11), and summing up the resulting inequalities, we obtain

$$\begin{aligned}\mu\|\mathbf{u}_H-\mathbf{u}_H^0\|_V^2\le& b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{u}_H^0-\mathbf{u}_H)+\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{u}_H^0-\mathbf{u}_H)\\ \le& N\|\mathbf{u}_H\|_V^2\|\mathbf{u}_H^0-\mathbf{u}_H\|_V+\lambda_1 H\|\mathbf{u}_H\|_V\|\mathbf{u}_H^0-\mathbf{u}_H\|_V\\ \le&\Big(\frac{\mu}{4}+\lambda_1 H\Big)\|\mathbf{u}_H\|_V\|\mathbf{u}_H^0-\mathbf{u}_H\|_V\\ \le&\frac{\kappa_1(\mu+4\lambda_1 H)}{4\mu+4\lambda_1 H}(\|\mathbf{f}\|+\|g\|_{L^2(S)})\|\mathbf{u}_H^0-\mathbf{u}_H\|_V,\end{aligned}$$

which completes the proof of (3.15a). For all $\mathbf{w}_H\in\mathbf{V}_{0H}$, taking $\mathbf{v}_H=\mathbf{u}_H\pm\mathbf{w}_H$ and $\mathbf{v}_H=\mathbf{u}_H^0\pm\mathbf{w}_H$ in the first inequalities of (3.4) and (3.11), and subtracting the resulting equations yield

$$d(\mathbf{w}_H,p_H-p_H^0)=a(\mathbf{u}_H-\mathbf{u}_H^0,\mathbf{w}_H)+b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{w}_H)+\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H,\mathbf{w}_H).$$

From (3.1), we get

$$\begin{aligned}\beta_1\|p_H-p_H^0\|\le&\mu]\|\mathbf{u}_H-\mathbf{u}_H^0\|_V+N\|\mathbf{u}_H\|_V^2+\lambda_1 H\|u_H\|_V\\ \le&\frac{2\kappa_1(\mu+4\lambda_1 H)}{4\mu+4\lambda_1 H}(\|\mathbf{f}\|+\|g\|_{L^2(S)}).\end{aligned}$$

So, we complete the proof. ☐

Lemma 3.2. *Under the uniqueness condition* (2.4), *let* $(\boldsymbol{u}_H,p_H)\in \boldsymbol{V}_H\times M_H$ *and* $(\boldsymbol{u}_H^n,p_H^n)\in \boldsymbol{V}_H\times M_H$ *be the solutions of* (3.4) *and* (3.12), *respectively. Then for* $n\in\mathbb{N}^+$, *we have*

$$\mu\|\boldsymbol{u}_H-\boldsymbol{u}_H^n\|_V\le\frac{\kappa_1(\mu+4\lambda_1 H)}{4\mu+4\lambda_1 H}\Big(\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}\Big)^n(\|\boldsymbol{f}\|+\|g\|_{L^2(S)})^{n+1},\tag{3.16a}$$

$$\beta_1\|p_H-p_H^n\|\le\frac{9\kappa_1(\mu+4\lambda_1 H)}{16\mu}\Big(\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}\Big)^n(\|\boldsymbol{f}\|+\|g\|_{L^2(S)})^{n+1}.\tag{3.16b}$$

Proof. Taking $(\mathbf{v}_H,q_H)=(\mathbf{u}_H^n,p_H^n-p_H)$ in (3.4) and $(\mathbf{v}_H,q_H)=(\mathbf{u}_H,p_H-p_H^n)$ in (3.12), and summing up the resulting inequalities yield

$$\begin{aligned}&a(\mathbf{u}_H^n-\mathbf{u}_H,\mathbf{u}_H^n-\mathbf{u}_H)+\frac{\lambda_1 H}{\mu}a(\mathbf{u}_H^n-\mathbf{u}_H,\mathbf{u}_H^n-\mathbf{u}_H)\\ \le& b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{u}_H^n-\mathbf{u}_H)-b(\mathbf{u}_H^{n-1},\mathbf{u}_H^n,\mathbf{u}_H^n-\mathbf{u}_H)\\ =&b(\mathbf{u}_H-\mathbf{u}_H^{n-1},\mathbf{u}_H,\mathbf{u}_H^n-\mathbf{u}_H).\end{aligned}$$

By using (2.1b) and (3.6), we derive

$$\begin{aligned}\|\mathbf{u}_H-\mathbf{u}_H^n\|_V\le&\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}(\|\mathbf{f}\|+\|g\|_{L^2(S)})\|\mathbf{u}_H-\mathbf{u}_H^{n-1}\|_V\\ \le&\Big(\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}(\|\mathbf{f}\|+\|g\|_{L^2(S)})\Big)^2\|\mathbf{u}_H-\mathbf{u}_H^{n-2}\|_V\le\cdots\\ \le&\Big(\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}(\|\mathbf{f}\|+\|g\|_{L^2(S)})\Big)^n\|\mathbf{u}_H-\mathbf{u}_H^{0}\|_V,\end{aligned}$$

which together with (3.15a) yields (3.16a). The estimate (3.16b) is directly from (3.1) and (3.16a). ☐

Remark 3.1. The uniqueness condition (2.4) implies

$$\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}(\|\mathbf{f}\|+\|g\|_{L^2(S)})<\frac{1}{4}.$$

Thus, the upper bounds for $\|\mathbf{u}_H-\mathbf{u}_H^n\|_V$ and $\|p_H-p_H^n\|$ can be revised to

$$\begin{aligned}\mu\|\mathbf{u}_H-\mathbf{u}_H^n\|_V\le&\frac{\kappa_1(\mu+4\lambda_1 H)}{4\mu+4\lambda_1 H}(\|\mathbf{f}\|+\|g\|_{L^2(S)})\Big(\frac{1}{4}\Big)^n,\\ \beta_1\|p_H-p_H^n\|\le&\frac{9\kappa_1(\mu+4\lambda_1 H)}{16\mu}(\|\mathbf{f}\|+\|g\|_{L^2(S)})\Big(\frac{1}{4}\Big)^n.\end{aligned}$$

As a direct consequence of Theorem 3.1 and Lemma 3.2, we derive the following error estimates for $(\mathbf{u}_H^n,p_H^n)$.

Theorem 3.2. *Under the uniqueness condition* (2.4)*, let* $(\boldsymbol{u},p)\in\boldsymbol{V}\times M$ *and* $(\boldsymbol{u}_H^n,p_H^n)\in\boldsymbol{V}_H\times M_H$ *be the solutions of* (2.2) *and* (3.12)*, respectively. Then for* $n\in\mathbb{N}^+$*, we have*

$$\begin{aligned}&\mu\|\boldsymbol{u}-\boldsymbol{u}_H^n\|_V+\|p-p_H^n\|\\ \le& C_3\Big(\inf_{\boldsymbol{v}_H\in\boldsymbol{V}_H}\mu\|\boldsymbol{u}-\boldsymbol{v}_H\|_V+\sqrt{\mu}\|\boldsymbol{u}_\tau-\boldsymbol{v}_{H\tau}\|_{L^2(S)}^{1/2}\\ &+\inf_{q_H\in M_H}\|p-q_H\|+\mu\lambda_1 H\Big)+M\Big(\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}\Big)^n(\|\boldsymbol{f}\|+\|g\|_{L^2(S)})^{n+1},\end{aligned}\tag{3.17}$$

where $M>0$ *depends on* μ, λ_1, β_1, κ_1, H. *Furthermore, if* $(\boldsymbol{u},p)\in\boldsymbol{H}^3(\Omega)\times H^2(\Omega)$*, then*

$$\mu\|\boldsymbol{u}-\boldsymbol{u}_H^n\|_V+\|p-p_H^n\|\le C_4(H^{5/4}+\lambda_1 H)+M\Big(\frac{N\kappa_1}{(\mu+\lambda_1 H)^2}\Big)^n(\|\boldsymbol{f}\|+\|g\|_{L^2(S)})^{n+1}.\tag{3.18}$$

3.2 Newton linearization on fine mesh

Based on the above discussion in Subsection 3.1, we replace $\mathbf{u}_H$ in (3.5) by $\mathbf{u}_H^n$. In this case, the problem (3.5) on the fine mesh is rewritten as

$$\begin{cases} \text{find } (\mathbf{u}_h,p_h)\in\mathbf{V}_h\times M_h \text{ such that} \\ a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)-d(\mathbf{v}_h-\mathbf{u}_h,p_h)+b(\mathbf{u}_H^n,\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)+b(\mathbf{u}_h,\mathbf{u}_H^n,\mathbf{v}_h-\mathbf{u}_h) \\ \quad +\dfrac{\lambda_2 h}{\mu}a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)+j(\mathbf{v}_{h\tau})-j(\mathbf{u}_{h\tau}) \\ \quad \geq(\mathbf{f},\mathbf{v}_h-\mathbf{u}_h)+b(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{v}_h-\mathbf{u}_h)+\dfrac{\lambda_2 h}{\mu}a(\mathbf{u}_H^n,\mathbf{v}_h-\mathbf{u}_h), \quad \forall\mathbf{v}_h\in\mathbf{V}_h, \\ d(\mathbf{u}_h,q_h)=0, \quad \forall q_h\in M_h. \end{cases} \tag{3.19}$$

Setting $(\mathbf{v}_h,q_h)=(2\mathbf{u}_h,p_h)$ and $(\mathbf{v}_h,q_h)=(0,p_h)$ in (3.19), we get

$$\begin{aligned} &a(\mathbf{u}_h,\mathbf{u}_h)+\frac{\lambda_2 h}{\mu}a(\mathbf{u}_h,\mathbf{u}_h)+b(\mathbf{u}_h,\mathbf{u}_H^n,\mathbf{u}_h) \\ =&(\mathbf{f},\mathbf{u}_h)-j(\mathbf{u}_{h\tau})+b(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{u}_h)+\frac{\lambda_2 h}{\mu}a(\mathbf{u}_H^n,\mathbf{u}_h). \end{aligned} \tag{3.20}$$

Under the uniqueness condition (2.4) and from (3.14), the left hand side of (3.20) satisfies

$$\begin{aligned} &a(\mathbf{u}_h,\mathbf{u}_h)+\frac{\lambda_2 h}{\mu}a(\mathbf{u}_h,\mathbf{u}_h)+b(\mathbf{u}_h,\mathbf{u}_H^n,\mathbf{u}_h) \\ \geq&(\mu+\lambda_2 h)\|\mathbf{u}_h\|_V^2-N\|\mathbf{u}_H^n\|_V\|\mathbf{u}_h\|_V^2 \\ \geq&\Big(\frac{3\mu}{4}+\lambda_2 h\Big)\|\mathbf{u}_h\|_V^2. \end{aligned}$$

The right hand side of (3.20) is bounded by

$$\begin{aligned} &(\mathbf{f},\mathbf{u}_h)-j(\mathbf{u}_{h\tau})+b(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{u}_h)+\frac{\lambda_2 h}{\mu}a(\mathbf{u}_H^n,\mathbf{u}_h) \\ \leq&\kappa_1(\|\mathbf{f}\|+\|g\|_{L^2(S)})\|\mathbf{u}_h\|_V+N\|\mathbf{u}_H^n\|_V^2\|\mathbf{u}_h\|_V+\lambda_2 h\|\mathbf{u}_H^n\|_V\|\mathbf{u}_h\|_V. \end{aligned}$$

Then we obtain

$$\begin{aligned} \|\mathbf{u}_h\|_V\leq&\frac{5\mu+4(\lambda_1 H+\lambda_2 h)}{(\mu+\lambda_1 H)(3\mu+4\lambda_2 h)}\kappa_1(\|\mathbf{f}\|+\|g\|_{L^2(S)}) \\ <&\frac{4\kappa_1}{\mu}(\|\mathbf{f}\|+\|g\|_{L^2(S)})<\frac{\mu}{N}. \end{aligned} \tag{3.21}$$

Next, we give the error estimates between the finite element approximation solution $(\mathbf{u}_h,p_h)$ on the fine mesh and the exact solution $(\mathbf{u},p)$.

Theorem 3.3. *Under the uniqueness condition* (2.4)*, let* $(\boldsymbol{u},p)\in \boldsymbol{V}\cap \boldsymbol{H}^3(\Omega)\times M\cap H^2(\Omega)$ *and* $(\boldsymbol{u}_h,p_h)\in \boldsymbol{V}_h\times M_h$ *be the solutions of* (2.2) *and* (3.19)*, respectively. Then there exist some* h_0*,* H_0*,* λ_{10}*,* λ_{20} *and* $n_0\in\mathbb{N}^+$ *such that when* $h<h_0$*,* $H<H_0$*,* $\lambda_1<\lambda_{10}$*,* $\lambda_2<\lambda_{20}$ *and* $n>n_0$ *there holds*

$$\mu\|\boldsymbol{u}-\boldsymbol{u}_h\|_V+\|p-p_h\|\le C_5(h^{5/4}+\|\boldsymbol{u}-\boldsymbol{u}_H^n\|_V^2+(\lambda_2+h)h\|\boldsymbol{u}-\boldsymbol{u}_H^n\|_V), \tag{3.22}$$

where $\boldsymbol{u}_H^n$ *is the solution of* (3.12).

Proof. Taking $\mathbf{v}=\mathbf{u}_h$ and $\mathbf{v}=2\mathbf{u}-\mathbf{v}_h$ in the first inequality of (2.2), respectively, we have

$$\begin{aligned}&a(\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)+b(\mathbf{u},\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)+j(\mathbf{u}_{h\tau})-2j(\mathbf{u}_\tau)+j(2\mathbf{u}_\tau-\mathbf{v}_{h\tau})-d(\mathbf{u}_h-\mathbf{v}_h,p)\\ &\ge(f,\mathbf{u}_h-\mathbf{v}_h),\end{aligned}$$

which together with (3.19) leads to

$$\begin{aligned}a(\mathbf{u}_h-\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)\le\,&b(\mathbf{u},\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)-b(\mathbf{u}_H^n,\mathbf{u}_h,\mathbf{u}_h-\mathbf{v}_h)-b(\mathbf{u}_h,\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)\\ &+b(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)+j(\mathbf{v}_{h\tau})-2j(\mathbf{u}_\tau)+j(2\mathbf{u}_\tau-\mathbf{v}_{h\tau})\\ &-d(\mathbf{u}_h-\mathbf{v}_h,p-p_h)-\frac{\lambda_2 h}{\mu}a(\mathbf{u}_h,\mathbf{u}_h-\mathbf{v}_h)+\frac{\lambda_2 h}{\mu}a(\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h).\end{aligned}$$

Thus, we get

$$\begin{aligned}\mu\|\mathbf{u}_h-\mathbf{v}_h\|_V^2\le\,&|a(\mathbf{u}-\mathbf{v}_h,\mathbf{u}_h-\mathbf{v}_h)|+|b(\mathbf{u},\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)-b(\mathbf{u}_H^n,\mathbf{u}_h,\mathbf{u}_h-\mathbf{v}_h)\\ &-b(\mathbf{u}_h,\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)+b(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)|\\ &+|j(\mathbf{v}_{h\tau})-2j(\mathbf{u}_\tau)+j(2\mathbf{u}_\tau-\mathbf{v}_{h\tau})|+|d(\mathbf{u}_h-\mathbf{v}_h,p-p_h)|\\ &+\Big|\frac{\lambda_2 h}{\mu}a(\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)-\frac{\lambda_2 h}{\mu}a(\mathbf{u}_h,\mathbf{u}_h-\mathbf{v}_h)\Big|\\ =\,&J_1+\cdots+J_5.\end{aligned} \tag{3.23}$$

By the similar arguments for I_1, I_3 and I_4 in the proof of Theorem 3.1, we can estimate J_1, J_3 and J_4, respectively, by

$$J_1\le\frac{\mu}{16}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+4\mu\|\mathbf{u}-\mathbf{v}_h\|_V^2,$$

and

$$J_3\le 2\|g\|_{L^2(S)}\|\mathbf{u}_\tau-\mathbf{v}_{h\tau}\|_{L^2(S)},$$

and

$$J_4\le\frac{\mu}{16}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\frac{4}{\mu}\|p-q_h\|^2+\varepsilon_2^2\|p_h-q_h\|^2+\frac{1}{4\varepsilon_2^2}\|\mathbf{u}-\mathbf{v}_h\|_V^2,$$

where ε_2 is some positive constant determined later. An alternative to J_2 is

$$\begin{aligned}J_2=&|\underbrace{b(\mathbf{u}_h-\mathbf{v}_h,\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)+b(\mathbf{v}_h-\mathbf{u},\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)}_{J_6}+\underbrace{b(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u},\mathbf{u}_h-\mathbf{v}_h)}_{J_7}\\&+\underbrace{b(\mathbf{v}_h-\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)}_{J_8}+\underbrace{b(\mathbf{u}_H^n-\mathbf{v}_h,\mathbf{v}_h-\mathbf{u}_H^n,\mathbf{u}_h-\mathbf{v}_h)}_{J_9}|.\end{aligned}$$

Then using (2.1b), (2.5), (3.18), (3.21) and Young inequality, we have

$$\begin{aligned}J_6\le& N\|\mathbf{u}\|_V\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+N\|\mathbf{u}\|_V\|\mathbf{u}-\mathbf{v}_h\|_V\|\mathbf{u}_h-\mathbf{v}_h\|_V\\\le&\frac{\mu}{4}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\frac{\mu}{16}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\frac{\mu}{4}\|\mathbf{u}-\mathbf{v}_h\|_V^2,\end{aligned}$$

and

$$J_7\le N\|\mathbf{u}_h\|_V\|\mathbf{u}-\mathbf{v}_h\|_V\|\mathbf{u}_h-\mathbf{v}_h\|_V\le\frac{\mu}{16}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+4\mu\|\mathbf{u}-\mathbf{v}_h\|_V^2,$$

and

$$\begin{aligned}J_8\le& N\|\mathbf{v}_h-\mathbf{u}_H^n\|_V\|\mathbf{u}_h-\mathbf{v}_h\|_V^2\\\le& N(\|\mathbf{u}-\mathbf{v}_h\|_V+\|\mathbf{u}-\mathbf{u}_H^n\|_V)\|\mathbf{u}_h-\mathbf{v}_h\|_V^2\\\le& C_6\Big(h^2+H^{5/4}+\lambda_1H+M\Big(\frac{N\kappa_1}{(\mu+\lambda_1H)^2}\Big)^n(\|\mathbf{f}\|+\|g\|_{L^2(S)})^{n+1}\Big)\|\mathbf{u}_h-\mathbf{v}_h\|_V^2\end{aligned}$$

with $C_6>0$ independent of h and H, and

$$J_9\le N\|\mathbf{u}_H^n-\mathbf{v}_h\|_V^2\|\mathbf{u}_h-\mathbf{v}_h\|_V\le\frac{\mu}{16}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\frac{4N^2}{\mu}\|\mathbf{v}_h-\mathbf{u}_H^n\|_V^4.$$

Finally, we estimate J_5 by

$$\begin{aligned}J_5=&\Big|\frac{\lambda_2h}{\mu}a(\mathbf{u}_H^n-\mathbf{v}_h,\mathbf{u}_h-\mathbf{v}_h)-\frac{\lambda_2h}{\mu}a(\mathbf{u}_h-\mathbf{v}_h,\mathbf{u}_h-\mathbf{v}_h)\Big|\\\le&\lambda_2h\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\lambda_2h\|\mathbf{u}_H^n-\mathbf{v}_h\|_V\|\mathbf{u}_h-\mathbf{v}_h\|_V\\\le&\lambda_2h\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\frac{\mu}{16}\|\mathbf{u}_h-\mathbf{v}_h\|_V^2+\frac{4\lambda_2^2h^2}{\mu}\|\mathbf{u}_H^n-\mathbf{v}_h\|_V^2.\end{aligned}$$

Combining these estimates into (3.23) and for sufficiently small h, H, λ_1, λ_2 and sufficiently large n such that

$$C_6\Big(h^2+H^{5/4}+\lambda_1H+M\Big(\frac{N\kappa_1}{(\mu+\lambda_1H)^2}\Big)^n(\|\mathbf{f}\|+\|g\|_{L^2(S)})^{n+1}\Big)+\lambda_2h<\frac{\mu}{16},$$

we get

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_V \le & C_7\Big(\|\mathbf{u}-\mathbf{v}_h\|_V+\frac{1}{\sqrt{\mu}}\|\mathbf{u}_\tau-\mathbf{v}_{h\tau}\|^{1/2}_{L^2(S)}+\frac{1}{\mu}\|p-q_h\|+\frac{1}{\mu}\|\mathbf{v}_h-\mathbf{u}^n_H\|^2_V \\ & +\frac{\lambda_2 h}{\mu}\|\mathbf{v}_h-\mathbf{u}^n_H\|_V\Big)+\frac{C_8\varepsilon_2}{\sqrt{\mu}}\|p_h-q_h\|+\frac{1}{2\varepsilon_2\sqrt{\mu}}\|\mathbf{u}-\mathbf{v}_h\|_V. \end{aligned} \tag{3.24}$$

For all $\mathbf{w}_h\in\mathbf{V}_{0h}$, we choose $\mathbf{v}=\mathbf{u}\pm\mathbf{w}_h$ in (2.2) and $\mathbf{v}_h=\mathbf{u}_h\pm\mathbf{w}_h$ in (3.19) and obtain

$$\begin{aligned}& a(\mathbf{u}-\mathbf{u}_h,\mathbf{w}_h)+b(\mathbf{u},\mathbf{u},\mathbf{w}_h)-b(\mathbf{u}^n_H,\mathbf{u}_h,\mathbf{w}_h)-b(\mathbf{u}_h,\mathbf{u}^n_H,\mathbf{w}_h) \\ & +b(\mathbf{u}^n_H,\mathbf{u}^n_H,\mathbf{w}_h)-\frac{\lambda_2 h}{\mu}a(\mathbf{u}_h-\mathbf{u}^n_H,\mathbf{w}_h)=d(\mathbf{w}_h,p-p_h).\end{aligned}$$

From (3.1) and

$$\begin{aligned}& b(\mathbf{u},\mathbf{u},\mathbf{w}_h)-b(\mathbf{u}^n_H,\mathbf{u}_h,\mathbf{w}_h)-b(\mathbf{u}_h,\mathbf{u}^n_H,\mathbf{w}_h)+b(\mathbf{u}^n_H,\mathbf{u}^n_H,\mathbf{w}_h) \\ =& b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},\mathbf{w}_h)+b(\mathbf{u},\mathbf{u}-\mathbf{v}_h,\mathbf{w}_h)-b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{v}_h,\mathbf{w}_h) \\ & +b(\mathbf{u}_h-\mathbf{u}^n_H,\mathbf{v}_h-\mathbf{u}^n_H,\mathbf{w}_h)-b(\mathbf{u}^n_H,\mathbf{u}_h-\mathbf{v}_h,\mathbf{w}_h) \\ \le & (N\|\mathbf{u}\|_V\|\mathbf{u}-\mathbf{u}_h\|_V+N\|\mathbf{u}\|_V\|\mathbf{u}-\mathbf{v}_h\|_V+N\|\mathbf{u}-\mathbf{v}_h\|_V\|\mathbf{u}-\mathbf{u}_h\|_V)\|\mathbf{w}_h\|_V \\ & +N(\|\mathbf{u}-\mathbf{u}_h\|_V+\|\mathbf{u}-\mathbf{u}^n_H\|_V)(\|\mathbf{u}-\mathbf{v}_h\|_V+\|\mathbf{u}-\mathbf{u}^n_H\|_V)\|\mathbf{w}_h\|_V \\ & +N\|\mathbf{u}^n_H\|_V(\|\mathbf{u}-\mathbf{u}_h\|_V+\|\mathbf{u}-\mathbf{v}_h\|_V)\|\mathbf{w}_h\|_V \\ \le & C_9(\mu+h^2+\|\mathbf{u}-\mathbf{u}^n_H\|_V)\|\mathbf{u}-\mathbf{u}_h\|_V\|\mathbf{w}_h\|_V+N\|\mathbf{u}-\mathbf{u}^n_H\|^2_V\|\mathbf{w}_h\|_V \\ & +N\|\mathbf{u}-\mathbf{u}^n_H\|_V\|\mathbf{u}-\mathbf{v}_h\|_V\|\mathbf{w}_h\|_V+C_{10}\mu\|\mathbf{u}-\mathbf{v}_h\|_V\|\mathbf{w}_h\|_V,\end{aligned}$$

we have

$$\begin{aligned}\beta_1\|p_h-q_h\|\le & \|p-q_h\|+C_9(\mu+\lambda_2 h+h^2+\|\mathbf{u}-\mathbf{u}^n_H\|_V)\|\mathbf{u}-\mathbf{u}_h\|_V+\lambda_2 h\|\mathbf{u}-\mathbf{u}^n_H\|_V \\ & +N\|\mathbf{u}-\mathbf{u}^n_H\|^2_V+N\|\mathbf{u}-\mathbf{u}^n_H\|_V\|\mathbf{u}-\mathbf{v}_h\|_V+C_{10}\mu\|\mathbf{u}-\mathbf{v}_h\|_V.\end{aligned} \tag{3.25}$$

For sufficiently small h, H, λ_1, λ_2 and sufficiently large n such that

$$\lambda_2 h+h^2+\|\mathbf{u}-\mathbf{u}^n_H\|_V<C_9\mu,$$

taking $\varepsilon_2=\frac{\beta_1}{4C_8C_9\sqrt{\mu}}$, and substituting the above estimate into (3.24), we get

$$\begin{aligned}\mu\|\mathbf{u}-\mathbf{u}_h\|_V \le & C_5(\|\mathbf{u}-\mathbf{v}_h\|_V+\|p-q_h\|+\|\mathbf{u}_\tau-\mathbf{v}_{h\tau}\|^{1/2}_{L^2(S)} \\ & +\|\mathbf{u}-\mathbf{u}^n_H\|^2_V+(\lambda_2+h)h\|\mathbf{u}-\mathbf{u}^n_H\|_V) \\ \le & C_5(h^{5/4}+\|\mathbf{u}-\mathbf{u}^n_H\|^2_V+(\lambda_2+h)h\|\mathbf{u}-\mathbf{u}^n_H\|_V).\end{aligned}$$

The estimate for pressure is immediately derived from (3.25). □

4 Numerical results

In this section, the numerical results are provided to confirm the convergence rates derived in Theorem 3.3. We implement all programs by the finite element software FreeFem++ [12]. We select the appropriate body force **f** such that the exact solution of (1.1) is of the following forms:

$$\mathbf{u}(x,y)=(u_1(x,y),u_2(x,y)), \qquad p(x,y)=(2x-1)(2y-1),$$
$$u_1(x,y)=-x^2y(x-1)(3y-2), \qquad u_2(x,y)=xy^2(y-1)(3x-2),$$

in the unit square $\Omega=(0,1)\times(0,1)$. A class of uniform triangular meshes of the unit square is made by the mesh generator in FreeFem++; see Fig. 1 for illustration.

It is easy to verify that the exact solution $\mathbf{u}$ satisfies $\mathbf{u}=0$ on $\Gamma=\{(x,y)|x=0,\ 0\le y\le 1\}\cup\{(x,y)|y=0,\ 0\le x\le 1\}$ and $\mathbf{u}_n=0$ on $S=S_1\cup S_2$, where $S_1=\{(x,y)|x=1,\ 0\le y\le 1\}$ and $S_2=\{(x,y)|y=1,\ 0\le x\le 1\}$. The tangential vector τ on S_1 and S_2 are $(0,1)$ and $(-1,0)$, respectively. Thus, we have

$$\begin{cases} \sigma_\tau=4\mu y^2(y-1) & \text{on } S_1, \\ \sigma_\tau=4\mu x^2(x-1) & \text{on } S_2. \end{cases}$$

On the other hand, from the friction slip boundary conditions (1.2), one has $|\sigma_\tau|\le g$. Therefore, the function g can be chosen as $g=-\sigma_\tau\ge 0$ on $S=S_1\cup S_2$.

Following Algorithm 4.1 in [16], we use the following Uzawa algorithms to solve the discrete variational inequality problems (3.11)-(3.12) and (3.19).

Step I: For

$$\lambda_H^0\in\Lambda_H \text{ is given}, \tag{4.1}$$

where Λ_H is defined in Section 3, then we solve the initial value $(\mathbf{u}_H^0,p_H^0)$ on the coarse

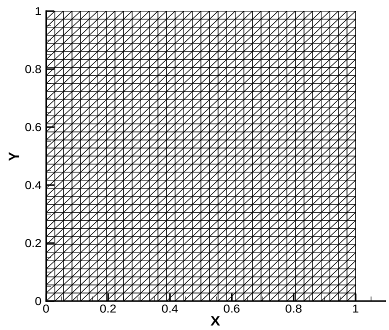

Figure 1: The FEM meshes of the unit square with $h=1/36$.

mesh by

$$\begin{cases} a(\mathbf{u}_H^0,\mathbf{v}_H)-d(\mathbf{v}_H,p_H^0)=(\mathbf{f},\mathbf{v}_H)-\int_S g\lambda_H^0\mathbf{v}_{H\tau}ds, & \forall\mathbf{v}_H\in\mathbf{V}_H,\\ d(\mathbf{u}_H^0,q_H)=0, & \forall q_H\in M_H. \end{cases} \tag{4.2}$$

Step II: We solve $(\mathbf{u}_H^n,p_H^n)\in\mathbf{V}_H\times M_H$ and λ_H^n with $n\in\mathbb{N}^+$ on the coarse mesh by

$$\begin{cases} a(\mathbf{u}_H^n,\mathbf{v}_H)-d(\mathbf{v}_H,p_H^n)+b(\mathbf{u}_H^{n-1},\mathbf{u}_H^n,\mathbf{v}_H)+\dfrac{\lambda_1 H}{\mu}a(\mathbf{u}_H^n,\mathbf{v}_H)=(\mathbf{f},\mathbf{v}_H)-\int_S g\lambda_H^{n-1}\mathbf{v}_{H\tau}ds,\\ d(\mathbf{u}_H^n,q_H)=0, \end{cases} \tag{4.3}$$

and

$$\lambda_H^n=P_{\Lambda_H}(\lambda_H^{n-1}+\rho g\mathbf{u}_{H\tau}^n),\quad \rho>0, \tag{4.4}$$

where P_{Λ_H} is defined in Section 3 and $\rho>0$ is a positive parameter. The stopping criterion used in Step II is to require $\|\mathbf{u}_H^n-\mathbf{u}_H^{n-1}\|$ to be less than 10^{-6}.

Step III: For

$$\lambda_h^0\in\Lambda_h \text{ is given,} \tag{4.5}$$

we solve $(\mathbf{u}_h^m,p_h^m)$ and λ_h^m with $m\in\mathbb{N}^+$ on the fine mesh by

$$\begin{cases} a(\mathbf{u}_h^m,\mathbf{v}_h)-d(\mathbf{v}_h,p_h^m)+b(\mathbf{u}_H^n,\mathbf{u}_h^m,\mathbf{v}_h)+b(\mathbf{u}_h^m,\mathbf{u}_H^n,\mathbf{v}_h)+\dfrac{\lambda_2 h}{\mu}a(\mathbf{u}_h^m,\mathbf{v}_h)\\ =(\mathbf{f},\mathbf{v}_h)-\int_S g\lambda_h^{m-1}\mathbf{v}_{h\tau}ds+b(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{v}_h)+\dfrac{\lambda_2 h}{\mu}a(\mathbf{u}_H^n,\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ d(\mathbf{u}_h^m,q_h)=0, & \forall q_h\in M_h, \end{cases} \tag{4.6}$$

and

$$\lambda_h^m=P_{\Lambda_h}(\lambda_h^{m-1}+\rho g\mathbf{u}_{h\tau}^m),\quad \rho>0. \tag{4.7}$$

The stopping criterion used in Step III is to require $\|\mathbf{u}_h^m-\mathbf{u}_h^{m-1}\|$ to be less than 10^{-6}.

To confirm the convergence rates derived in Theorem 3.3, we take λ_1, λ_2, H and h to satisfy $H=\mathcal{O}(h^{1/2})$, $\lambda_1=\mathcal{O}(H^{1/4})$ and $\lambda_2=\mathcal{O}(1)$. Then we have

$$\mu\|\mathbf{u}-\mathbf{u}_h\|_V+\|p-p_h\|\le Ch^{5/4}. \tag{4.8}$$

Thus, in the numerical experiments, we choose $h=H^2$, $\lambda_1=0.1\times H^{1/4}$, $\lambda_2=0.1$ for six coarse meshes $H=1/2i$, $i=2,\cdots,7$. On the other hand, the parameter ρ in Uzawa algorithms (4.4) and (4.7) is taken as $\rho=0.5\mu$ for differen viscosity $\mu=1/Re$. Table 1 displays the numerical results for $Re=1000$. The convergence rates for the velocity in $\mathbf{H}^1$ norm and the pressure in L^2 norm are shown in Fig. 2, from which we can see that the these two convergence rates both reach the theoretical rate derived in (4.8). For fixed $H=0.1$

Table 1: Numerical errors for different meshes with $Re=1000$.

$1/H$	$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_V}{\|\mathbf{u}\|_V}$	$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}\|}$	$\frac{\|p-p_h\|}{\|p\|}$	CPU (s)
4	4^2	3.73719×10^{-1}	1.94203×10^{-2}	3.02440×10^{-3}	19.715
6	6^2	5.28079×10^{-2}	1.21062×10^{-3}	5.97626×10^{-4}	51.300
8	8^2	1.27585×10^{-2}	1.66902×10^{-4}	1.89105×10^{-4}	104.327
10	10^2	4.20740×10^{-3}	3.78787×10^{-5}	7.74590×10^{-5}	202.252
12	12^2	1.69559×10^{-3}	1.40288×10^{-5}	3.74753×10^{-5}	338.955
14	14^2	7.86056×10^{-4}	5.10885×10^{-6}	2.02664×10^{-5}	627.433

Table 2: Numerical errors for different Reynolds numbers with $H=0.1$ and $h=0.01$.

Re	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_V}{\|\mathbf{u}\|_V}$	$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}\|}$	$\frac{\|p-p_h\|}{\|p\|}$	CPU (s)
1000	4.20740×10^{-3}	3.78787×10^{-5}	7.74590×10^{-5}	202.252
2000	8.33878×10^{-3}	7.59606×10^{-5}	7.74593×10^{-5}	327.045
3000	1.23476×10^{-2}	1.10744×10^{-4}	7.74596×10^{-5}	439.783
4000	1.62023×10^{-2}	1.51006×10^{-4}	7.74607×10^{-5}	544.076
5000	1.98922×10^{-2}	1.83698×10^{-4}	7.74614×10^{-5}	670.219
6000	2.34221×10^{-2}	2.28958×10^{-4}	7.74642×10^{-5}	759.349
7000	2.68025×10^{-2}	2.49555×10^{-4}	7.74638×10^{-5}	853.428
8000	3.00486×10^{-2}	2.84513×10^{-4}	7.74660×10^{-5}	944.610
9000	3.31741×10^{-2}	3.24598×10^{-4}	7.74694×10^{-5}	1194.11
10000	3.61920×10^{-2}	3.56464×10^{-4}	7.74715×10^{-5}	1323.12
Rate	0.9013	0.9607	0	/

and $h=0.01$, we provide the numerical results for different Reynolds numbers $Re=1000i$, $i=1,\cdots,10$ in Table 2, from which we can see that the $\mathbf{H}^1$ errors for velocity and L^2 errors for pressure are of the first order and the zero order with respect to Re, respectively. These numerical results are in good agreement with (4.8). Finally, we show the contour plots of exact and numerical velocity and pressure to exhibit the approximation profiles in details. Figs. 3-5 present the exact solution, the numerical solutions with $Re=5000$

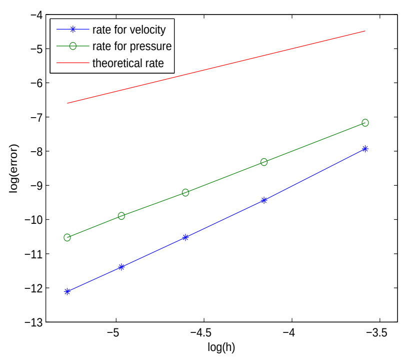

Figure 2: Convergence rates of velocity and pressure.

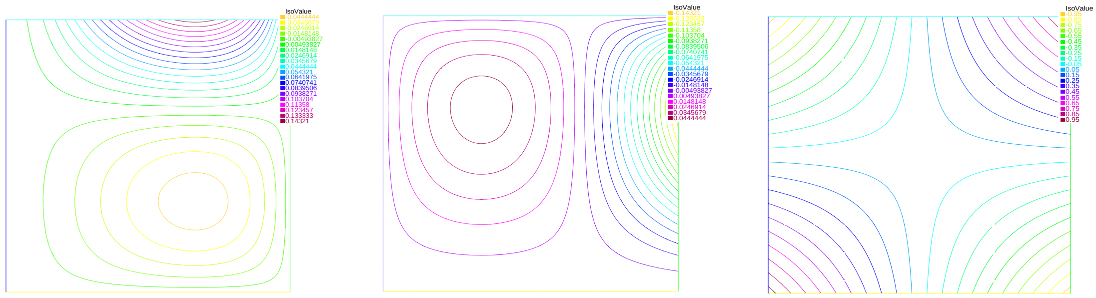

Figure 3: Contour plots of exact solution. From left to right: two components of velocity and pressure.

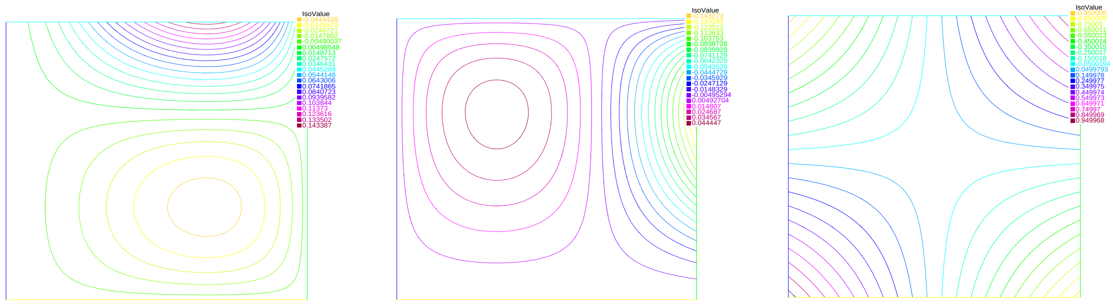

Figure 4: Contour plots of numerical solution with $Re=5000$, $H=0.1$, $h=0.01$. From left to right: two components of velocity and pressure.

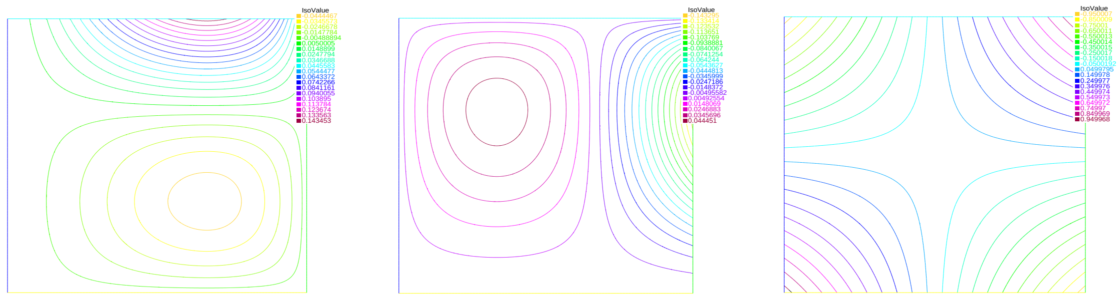

Figure 5: Contour plots of numerical solution with $Re=10000$, $H=0.1$, $h=0.01$. From left to right: two components of velocity and pressure.

and 10000, respectively. From these three groups of contour plots, we can observe the good coincidence with each other to illustrate the efficiency and stability of the present two-level defect-correction method.

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