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Two-Level Defect-Correction Method for Steady Navier-Stokes Problem with Friction Boundary Conditions

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Abstract. In this paper, we present two-level defect-correction finite element method for steady Navier-Stokes equations at high Reynolds number with the friction boundary conditions, which results in a variational inequality problem of the second kind. Based on Taylor-Hood element, we solve a variational inequality problem of Navier-Stokes type on the coarse mesh and solve a variational inequality problem of Navier-Stokes type corresponding to Newton linearization on the fine mesh. The error estimates for the velocity in the $\mathbf{H}^1$ norm and the pressure in the L^2 norm are derived. Finally, the numerical results are provided to confirm our theoretical analysis.

AMS subject classifications: 65N30, 76M10

Key words: Navier-Stokes equations, friction boundary conditions, variational inequality problems, defect-correction method, two-level mesh method.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and convex domain with Lipschitz boundary $\partial\Omega$. Consider steady incompressible flows which are governed by:

$$\begin{cases} -\mu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla p=\mathbf{f} & \text{in } \Omega, \\ \text{div } \mathbf{u}=0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u}=(u_1,u_2)$ denotes the velocity vector of the flows, p the pressure and $\mathbf{f}=(f_1,f_2)$ the body force vector. The constant $\mu=1/Re>0$ is the viscosity with Reynolds number Re. In this paper, the following friction boundary conditions are considered:

$$\begin{cases} \mathbf{u}=0 & \text{on } \Gamma, \\ \mathbf{u}_\mathbf{n}=\mathbf{u}\cdot\mathbf{n}=0, \quad -\sigma_\tau(\mathbf{u})\in g\partial|\mathbf{u}_\tau| & \text{on } S, \end{cases} \tag{1.2}$$

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where $\Gamma\cap S=\emptyset$ and $\overline{\Gamma\cup S}=\partial\Omega$. g is a scalar function. $\mathbf{n}$ represents the unit vector of the external normal to S. $\mathbf{u}_\tau$ and $\sigma_\tau(\mathbf{u})$ are the tangential components of the velocity and the stress vector σ which is defined by $\sigma_i=\sigma_i(\mathbf{u},p)=(\mu e_{ij}(\mathbf{u})-p\delta_{ij})n_j$ with $e_{ij}(\mathbf{u})=\frac{\partial u_i}{\partial x^j}+\frac{\partial u_j}{\partial x^i}$, $i,j=1,2$. The subdifferential set is defined as follows. Let ψ be a given function which is of convexity and weak semi-continuity from below. The subdifferentialset $\partial\psi(a)$ is defined by

$$\partial\psi(a)=\{b\in\mathbb{R}:\psi(h)-\psi(a)\geq b(h-a),\ \ \forall h\in\mathbb{R}\}.$$

The boundary conditions (1.2) were introduced by H. Fujita to describe some problems in hydrodynamics [5]. Some well-posedness results from the view of theory have been studied, such as R. An, Y. Li and K. Li [1], H. Fujita [6–8], T. Kashiwabara [15], Y. Li and K. Li [26, 28], Le Roux [31, 32], N. Saito [33] and references cited therein. Although there are a large amount of works about the finite element methods for Navier-Stokes equations, however, the numerical methods for the problem (1.1)-(1.2) have not been studied as much. The reason is that the variational formulation of (1.1)-(1.2) is of the form of variational inequality due to the subdifferential property on the boundary S. M. Ayadi, M. Gdoura and T. Sassi studied mini-element method for Stokes problem in [3]. T.Kashiwabara studied optimal finite element error bounds by defining the different numerical integration of the non-differential term on the boundary S corresponding to the different finite element pairs [16, 17]. The penalty and stabilized finite element methods and their two-level mesh methods for steady problem were studied in [2, 4, 22–25]. In these works, all numerical experiments were displayed only for small Reynolds number. It is well known that for the incompressible flows at high Reynolds number, Navier-Stokes equations are the domination of the convection and the flows are very unstable. Thus, it is difficult to make the numerical simulation of the incompressible flows efficiently.

There are some stabilized methods to overcome the difficulty in simulating the incompressible flows at high Reynolds number numerically, such as the variational multiscale method [13, 14, 34, 35], the subgrid method [11, 21], the defect-correction method [18–20, 29], etc. The defect-correction method is an iterative improvement technique and can increase the accuracy of the solution without refining the mesh, so it has been successively applied to Navier-Stokes equations at high Reynolds number. W. Layton firstly studied defect-correction method for the steady incompressible flows at high Reynolds number in [19]. Recently, H. Qiu and L. Mei studied the two-level defect-correction method for steady Navier-Stokes problem by using the stabilized finite element method [30].

In this paper, we combine defect-correction method with two-level mesh technique to solve the problem (1.1) at high Reynolds number with friction boundary conditions (1.2) numerically. Since the variational formulation of the problem (1.1)-(1.2) is the variational inequality problem, there exist some differences between our method for the problem (1.1)-(1.2) and those for Navier-Stokes equations (1.1) with Dirichlet boundary conditions. The main idea of our two-level method is to solve a nonlinear variational inequality problem of Navier-Stokes type at the defect step on the coarse mesh and solve a lin-