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On a Large Time-Stepping Method for the Swift-Hohenberg Equation

Zhengru Zhang[1,*] and Yuanzi Ma[2]

[1] *Laboratory of Mathematics and Complex Systems, Ministry of Education and School of Mathematical Sciences, Beijing Normal University, 19 Xinjiekouwai Street, Haidian District, Beijing 100875, China*
[2] *College of Science, China University of Petroleum–Beijing, 18 Fuxue Road, Changping, Beijing 102249, China*

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Abstract. The main purpose of this work is to contrast and analyze a large time-stepping numerical method for the Swift-Hohenberg (SH) equation. This model requires very large time simulation to reach steady state, so developing a large time step algorithm becomes necessary to improve the computational efficiency. In this paper, a semi-implicit Euler schemes in time is adopted. An extra artificial term is added to the discretized system in order to preserve the energy stability unconditionally. The stability property is proved rigorously based on an energy approach. Numerical experiments are used to demonstrate the effectiveness of the large time-stepping approaches by comparing with the classical scheme.

AMS subject classifications: 65M06, 65Z05

Key words: Large time-stepping method, energy stable, Swift-Hohenberg equation, finite difference method.

1 Introduction

There have been many dynamical models of non-equilibrium system which often results in highly complicated domain structures. There are many well-known models including Cahn-Hilliard equation (for conservative system) [2] and Allen-Cahn model (for non-conservative system) [1] which are used to describe the phase interface, molecular beam epitaxy growth model [11] which is used to describe the height of the thin solid films, the phase field crystal (PFC) equation which is used to describe the defects in crystal. The Swift-Hohenberg (SH) model [10] is a non-conserved form of the PFC equation. Recently, these models are developed to couple with incompressible fluids [3, 14], and the corresponding energy laws are derived.

*Corresponding author.
Email: zrzhang@bnu.edu.cn (Z. R. Zhang), yzma9017@126.com (Y. Z. Ma)

The SH model can account for elastic and plastic deformations of the lattice, dislocations, grain boundaries, and many other observable phenomena, see [12]. The models that are minimized by periodic functions were accounted for the periodic structure of a crystal lattice as free energy functionals. This model differs from the CH and AC models in that the stable phase is periodic, see [4]. For SH models, the order parameter is viewed as capturing the inhomogeneities in a fluid associated with Rayleigh-Bénard convection. The theory of the SH problem has been studied for decades. However, both the fourth-order and the nonlinear term make the SH equation stiff and difficult to solve numerically.

The standard Euler integration is known to be unstable for time step Δt above a threshold fixed by lattice spacing Δx [9]. In CH and AC systems, to maintain an interfacial profile, the lattice spacing must be smaller than the interfacial width. As we know for these models such as CH or AC equation, molecular beam epitaxy model and SH model, it takes large time to reach steady state. So developing large time stepping method becomes necessary. The large time stepping idea proposed by Xu and Tang [13] for the molecular beam epitaxy simulation allows rather larger time step than the classical time discretizations, and has been used to solve CH equation by He and Tang [6]. Another method to avoid small time step is to use adaptive time step strategy, which is successfully applied to molecular beam epitaxy model [8] and CH model [15]. The time steps are chosen based on the variation of the energy functional with respect to time variable. In SH system, the time step Δx must be smaller than the periodicity selected by the system. In this paper, we will apply the similar idea in [13] to the simulation of SH model. A large time-stepping method for the SH equation is proposed with an extra term added to the classical Euler approach, which is consistent with the order of the time discretization. The fourth order term is treated implicitly and the concave part is approximated explicitly. The proposed difference scheme is proved to preserve energy stability unconditionally. In this work the energy means the free energy functional corresponding to the SH model in stead of the traditional L^2-norm as discussed in our recent work for epitaxial growth model [7].

This paper is organized as below. In Section 2, the SH model is introduced and the numerical scheme is proposed. In Section 3, we establish the stability analysis about the numerical algorithm. Numerical experiments are presented in Section 4 to show the effectiveness of the proposed scheme. Finally, in Section 5, some concluding remarks are made.

2 The SH model and the unconditionally stable scheme

Both the SH and PFC models come from minimizing the free energy functional that describes the configurational cost of periodic phase in contact with isotropic phase [5, 10]:

$$E(\phi)=\int_\Omega \left\{\frac{1}{4}\phi^4+\frac{1-\epsilon}{2}\phi-|\nabla\phi|^2+\frac{1}{2}(\Delta\phi)^2\right\}dx, \tag{2.1}$$