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A Unified Instability Region for the Extended Taylor–Goldstein Problem of Hydrodynamic Stability

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Abstract. We consider inviscid, incompressible shear flows with variable density and variable cross section. For this problem, we derived a new estimate for the growth rate of an unstable mode and a parabolic instability region which intersects semiellipse instability region under some condition.

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1 Introduction

The study of inviscid, incompressible shear flows in sea straits with variable density and variable cross section was initiated by [1] and [2] developed the mathematical approach. In the above work, if density remains constant then it leads to extended Rayleigh problem of hydrodynamic stability, while the variable density leads to the extended Taylor-Goldstein problem of hydrodynamic stability. In our present work, we consider extended Taylor-Goldstein problem. For this extended Taylor-Goldstein problem, many general analytical results have been obtained in [2–7].

In [3], they defined the function

$$\phi(z)=\frac{U_0''-TU_0'+2\pi^2[U_{0\max}-U_{0\min}]b_{\min}}{D^2b_{\max}}>0,$$
$$\psi(z)=\frac{U_0''-TU_0'-2\pi^2[U_{0\max}-U_{0\min}]b_{\min}}{D^2b_{\max}}<0.$$

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For flows satisfying $\phi(z)>0$ or $\psi(z)<0$, the semiellipse instability region of [3] is reduced. [4] extended their work and derived a parabolic instability region which depends $U_{0\min}>0$. In the present paper, we have obtained a new estimate for the growth rate of an unstable mode and a parabolic instability region which intersects semiellipse instability region given in [3] without any conditions as given in [3,4]. We illustrate our results with examples.

2 Extended Taylor–Goldstein problem

The extended Taylor–Goldstein Problem [2] is an eigen value problem given by

$$\left[\frac{(bW)'}{b}\right]'+\left[\frac{N^2}{(U_0-c)^2}-\frac{b(\frac{U_0'}{b})'}{U_0-c}-k^2\right]W=0, \tag{2.1}$$

with boundary conditions

$$W(0)=0=W(D). \tag{2.2}$$

Here W is the complex eigen function, $c=c_r+ic_i$ is the complex phase velocity, $N^2>0$ is the Brunt Vaisala frequency, $k>0$ is the wave number, U_0 is the basic velocity profile, $b(z)$ is the breadth function.

3 Estimate for growth rate

Theorem 3.1. *An estimate for the growth rate of an unstable mode is given by*

$$k^2c_i^2\le\left[\frac{\left[\frac{|N^2_{\max}|}{2}-|N^2_{\min}|\right]+|b(\frac{U_0'}{b})'|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{4}\right]}{\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}k^2}+1\right]}\right].$$

Proof. Multiplying (2.1) by (bW^*), where $*$ stands for complex conjugation; integrating over $[0,D]$ and using (2.2), we get

$$\int\frac{|(bW)'|^2}{b}dz+\int\frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c)}b|W|^2dz+k^2\int b|W|^2dz-\int\frac{N^2}{(U_0-c)^2}b|W|^2dz=0. \tag{3.1}$$

Equating real part of (3.1), we get

$$\begin{aligned}&\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz+\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)b|W|^2dz\\&-\int\frac{N^2}{|U_0-c|^4}[(U_0-c_r)^2-c_i^2]b|W|^2dz=0.\end{aligned} \tag{3.2}$$