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An Extrapolation Cascadic Multigrid Method for Elliptic Problems on Reentrant Domains

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Abstract. This paper proposes an extrapolation cascadic multigrid (EXCMG) method to solve elliptic problems in domains with reentrant corners. On a class of λ-graded meshes, we derive some new extrapolation formulas to construct a high-order approximation to the finite element solution on the next finer mesh using the numerical solutions on two-level of grids (current and previous grids). Then, this high-order approximation is used as the initial guess to reduce computational cost of the conjugate gradient method. Recursive application of this idea results in the EXCMG method proposed in this paper. Finally, numerical results for a crack problem and an L-shaped problem are presented to verify the efficiency and effectiveness of the proposed EXCMG method.

AMS subject classifications: 65N55, 65N30

Key words: Richardson extrapolation, Cascadic multigrid, graded mesh, elliptic problems, corner singularity.

1 Introduction

It is well known that when an elliptic boundary value problem is solved by the finite element (FE) method on a quasi-uniform grid, the convergence rate is determined by the regularity of the solution [18]. Solutions of elliptic boundary value problems on domains with reentrant corners have singular behavior near the corners. This occurs even when

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the data of the underlying problem are very smooth. Such singular behavior significantly affects the accuracy of the FE method throughout the whole domain. For simplicity, we consider the Poisson equations with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a given function in the $L^2(\Omega)$ and Ω is an open, bounded domain in $\mathbb{R}^2$ with at least one reentrant corner.

The unique solution of problem (1.1) belongs to $H^{1+\pi/\omega-\epsilon}$ for any $\epsilon>0$ where ω is the maximum of the reentrant angles, and the standard continuous piecewise linear FE on a quasi-uniform grid yields $\mathcal{O}(h^{\pi/\omega-\epsilon})$ and $\mathcal{O}(h^{2\pi/\omega-\epsilon})$ accuracy in the H^1 and the L^2 norms, respectively. There are several approaches (such as conformal transformation methods, local mesh refinement and singular function methods) in the literature to overcome this difficulty (see [1–3,7–12,19] and references therein).

Locally refined grid for singular solutions was first studied by Babuska [1,2]. Then, Schatz and Wahlbin [34] introduced a more general, locally refined grid to handle singularity. With this grid, optimal order convergence rate can be obtained. Huang and Lin [28,29] proposed a radial shrinkage transformation method, which greatly simplifies the error analysis of FE approximations. The advantage of the local mesh refinement is that the knowledge of the exact form of the singular functions is not needed, so it becomes one of the most important methods to solve singular problems. However, when the singularity of solutions is intensive or a high accuracy is required, the number of grid points increases significantly. In these circumstance, classical iterative methods fail to be effective.

The multigrid (MG) method is regarded as one of the most effective methods for solving Poisson problems. In [27], Huang and Mu studied the extrapolation and MG algorithms for stress intensity factors on reentrant domains. Brenner [7–9] successfully applied the classical MG methods to compute the regular part of singular solutions on corner domains. Cai and Kim developed a new FE method based MG solvers by using singular functions for the Poisson equation on a polygonal domain with reentrant angles [10,11]. However, classical MG methods have to cycle between coarse and fine grids to accelerate their rate of convergence. The cascadic multigrid (CMG) method proposed in 1996 by Deuflhard and Bornemann is a simpler multi-level method without coarse-grid correction [5]. Since then, many scholars have carried out extensive studies on the theoretical analysis and application of the method [6,35,36,38,39]. In 2008, we proposed an extrapolation cascadic multigrid method (EXCMG) for solving elliptic boundary value problems [14,15]. The algorithm is based on the idea of CMG, while the linear interpolation on the coarse grid is modified as extrapolation and quadratic interpolation, which enable us to obtain a better initial guess for the iterative solution on the next finer grid. Then the conjugate gradient (CG) method is used to solve the resulting linear system with the good initial guess. After several years' development, the EXCMG algorithm has been successfully applied to non-smooth problems [22], parabolic problems [21], and

some other related problems [17,25,26,30,31]. Recently, Pan et al. further improved and generalized the EXCMG method to solve three dimensional (3D) elliptic boundary value problems [32], obtained the super-optimality of the EXCMG method under the energy norm for $H^{2+\alpha}$-regular $(0<\alpha\leq 1)$ problems in both two and three dimensions [23], and proposed an EXCMG method to solve 3D Poisson equation combined with a fourth-order compact difference scheme [33].

The λ-graded grid [13] can not only ensure optimal convergence rate of FE solutions, but also maintain the superconvergence and extrapolation. Furthermore, the generation of the graded grid is trivial. The goal of this paper is to extend the EXCMG method to elliptic problems on domains with reentrant corners by using piecewise λ-graded grid.

The rest of the paper is organized as follows. Section 2 contains singularities on reentrant domains and the weighted regularity estimate of singular problems. In Section 3, we presents some error estimates of FE approximations based on λ-graded grid. Some new extrapolation formulas and the EXCMG algorithm for elliptic problems on reentrant domains are introduced in Section 4. Numerical examples are provided to verify the effectivity of the method in Section 5. And conclusions are given in the final section.

In the following discussion, the symbol C denotes a generic positive constant which may vary with the context, but is always independent of the mesh size.

2 Singularity on reentrant domains

Suppose that the maximum interior angle on the domains is $\omega=\alpha\pi$, $1<\alpha<2$. Without loss of generality, assume that the corresponding vertex is at the origin. Let polar coordinates (r,θ) be chosen at the origin so that the internal angle ω is spanned by the two half lines $\theta=0$ and $\theta=\omega$. Then the unique solution of problem (1.1) has the singular function representation of the form

$$u=w+\lambda r^{\frac{1}{\alpha}}\sin\Big(\frac{\theta}{\alpha}\Big), \tag{2.1}$$

where $\lambda\in\mathbb{R}$ is the so-called *stress intensity factor* and $w\in H^2(\Omega)$ is the regular part of the solution. Moreover, the following regularity estimate holds:

$$|D^k u|\leq Cr^{\frac{1}{\alpha}-|k|}. \tag{2.2}$$

It is well known that the solution u of the problem (1.1) is in $H^{1+1/\alpha-\epsilon}(\Omega)$ for any $\epsilon>0$. For a crack domain ($\alpha=2$), the solution merely belongs to $H^{1.5-\epsilon}$, see [20] for details. Such lack of regularity affects the accuracy of the FE approximation.

When the problem (1.1) is solved by linear FE method on a quasi-uniform grid, we have the following error estimates [18]:

$$\|u-u_h\|=\mathcal{O}(h^{2\gamma}) \tag{2.3}$$

and

$$\|u-u_h\|_1=\mathcal{O}(h^{\gamma}), \tag{2.4}$$

for $\gamma=\frac{1}{\alpha}-\epsilon<1$. The error estimate (2.3) will be verified in Table 1 and Table 5 in the numerical results. However, optimal convergence rates $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$ cannot be obtained.

It is well known that the success of the extrapolation technique relies on the existence of asymptotic error expansions. Huang [24] and Blum [4] studies the extrapolation method for the numerical solution of elliptic boundary value problems on reentrant domains. The main result of above mentioned two papers is presented in the following theorem.

Theorem 2.1 (see [4,24]). *Assume Ω is a polygon, and ω_j $(1\leq j\leq n)$ are its internal angels. Let u^h be the linear FE solution to the Dirichlet problem for Poisson's equation. Under the assumption on the smoothness of the given data, at every nodal point x, there is an asymptotic error expansion,*

$$u^h(x)=u(x)+\sum_{j=1}^{n}A_j(x)h^{2\pi/\omega_j}+\mathcal{O}(h^2|\log h|), \tag{2.5}$$

which holds on the quasi-uniform mesh with mesh size h. Here, n is the number of reentrant corners, $A_j(x)$ are suitable smooth functions.

3 Weighted error estimate on λ-graded grid

3.1 λ-graded grid

Consider the graded grid on a sector domain Ω, the step size is set as $h=1/N$, and grid points are set on radius vector r:

$$r_j=(jh)^\lambda,\quad j=0,1,2,\cdots,N. \tag{3.1}$$

Several arcs S_j are drawn and each S_j is divided into j equal parts. By connecting these partition points using straight line segments, a λ-graded grid is generated (see Fig. 1). Obviously, when $\lambda>1$, there is a locally refined grid at origin O, and for element τ_j on the j-th layer between arcs S_{j-1} and S_j, the step size along the radial direction is

$$h_j=r_j-r_{j-1}=(jh)^\lambda\left(1-\left(1-\frac{1}{j}\right)^\lambda\right)\approx\lambda r_j/j=\lambda h r_j^{1-1/\lambda}.$$

The step size of element τ_1 on the first layer near to the origin O is the smallest, that is, $h_1=h$; while the step size of the element τ_N close to the unit arc is $h_N\approx\lambda h$. The ratio of two adjacent points r_{j+1} to r_j is $(r_j+1)/r_j=(1+1/j)\lambda\leq 2\lambda$, which is uniformly bounded. Then the step size difference between two adjacent elements is

$$h_{j+1}-h_j=(jh)^\lambda\left(\left(1+\frac{1}{j}\right)^\lambda+\left(1-\frac{1}{j}\right)^\lambda-2\right)\approx\lambda(\lambda-1)\frac{r_j}{j^2}\approx(\lambda-1)\frac{h_j}{j}. \tag{3.2}$$

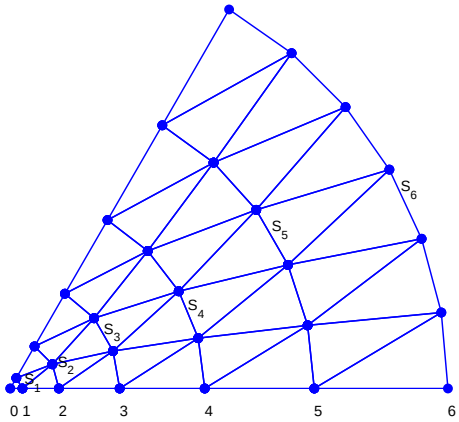


Figure 1: λ-graded grid on a sector $(\lambda=2,\ N=6)$.

3.2 Weighted error estimate

Weighted regularity estimate can be obtained by using the weighted space proposed in [13]. The solution of (1.1) shows the following weighted regularity estimate:

$$\|r^s D^2 u\| + \|r^{s-1} Du\| + \|r^{s-2} u\| \leq C\|r^s f\| \tag{3.3}$$

for $1-\frac{1}{\alpha} < s < 1+\frac{1}{\alpha}$.

It should be noted that λ-graded grid can automatically generate the required weight function. Based on such graded grid, the following error estimations of linear FE approximation hold.

Theorem 3.1 (see [13]). *Suppose that sector Ω contains central angle $\alpha\pi$ and a λ-graded grid is generated in Ω. Let u be the solution of boundary problem* (1.1) *and u_h be the linear FE approximation. If $\lambda > \alpha$, then the following optimal order error estimates hold:*

$$\|u-u_h\| \leq Ch^2 \|r^s D^2 u\| \tag{3.4}$$

and

$$\|u-u_h\|_1 \leq Ch\|r^s D^2 u\|. \tag{3.5}$$

4 EXCMG based on a graded grid

4.1 Extrapolation for true solution

Suppose that u is the exact solution of the differential equation, and u_h is the linear (or bilinear) FE solution on grid Z_h. Denote $e^h = u - u^h$ and $e_k^h = e^h(x_k)$. For elliptic problems in domains with reentrant corners, we already have error estimates of FE approximations

(2.3), (3.4) and (3.5) in the L^2 norm, and further assume that the truncation error at node x has the form

$$e^h(x)=A(x)h^\alpha+\mathcal{O}(h^\beta),\quad 0<\alpha<\beta, \tag{4.1}$$

where $A(x)$ is the suitably smooth function. When we use quasi-uniform meshes, we already have the asymptotic error expansion (2.5). For the L-shape domain, $\alpha=4/3$ and $\beta=2$, see [24] for details.

If the grid is refined, a grid $Z_{h/2}$ with step size $h/2$ is obtained. On each node x, the FE solution $u^{h/2}$ shows:

$$e^{h/2}(x)=A(x)h^\alpha/2^\alpha+\mathcal{O}(h^\beta). \tag{4.2}$$

Then on grid point x_k, $A(x_k)$ on the right-hand side of the equation can be easily eliminated to obtain a high-precision extrapolation formula,

$$Eu_k^{h/2}:=\frac{2^\alpha u_j^{h/2}-u_k^h}{2^\alpha-1}=u_k+\mathcal{O}(h^\beta),\quad k=j,j+1. \tag{4.3}$$

Some scholars have applied the extrapolation formula in MG, although the results are unsatisfactory. The authors found that the extrapolation value of the above formula was excessively accurate when it was used as the initial guess of the iterative solution on the next finer grid. This is because the FE solution on next finer grid $Z_{h/4}$ should have an error $A(x_j)h^\alpha/4^\alpha$. If an over-accurate initial guess with error $\mathcal{O}(h^\beta)$ is adopted, additional iterations need to be performed. Such iterations play a negligible role.

4.2 Extrapolation for FE solution

A new extrapolation formula is designed to approximate the FE solution $u_{h/4}$, rather than the exact solution u.

Taking a one-dimensional grid as an example, for the three embedded grids Z_i with mesh size $h_i=h_0/2^i$, $i=0,1,2$ in Fig. 2, the corresponding linear FE solution is u_i. The coarse element $\tau_{j+1}=(x_j^0,x_{j+1}^0)$ of Z_0 is refined to two elements of Z_1 and four elements

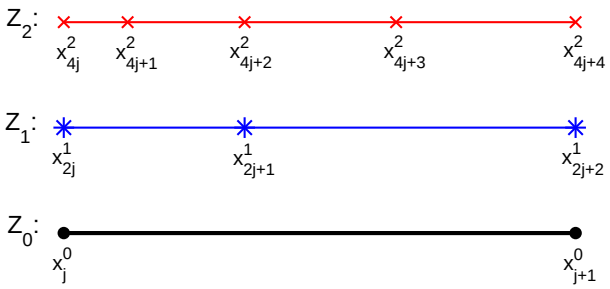


Figure 2: Three levels of embedded graded grid ($\lambda=2$).

of Z_2. By adding a midpoint and two quartiles (when the local refinement index $\lambda>1$, they are not actual midpoints and quartiles), a set of five points is obtained:

$$\{x^2_{4j},\, x^2_{4j+1},\, x^2_{4j+2},\, x^2_{4j+3},\, x^2_{4j+4}\}.$$

Suppose that the FE solutions on two groups of nodes are known:

$$Z_0\colon \{u^0_j,\, u^0_{j+1}\}; \qquad Z_1\colon \{u^1_{2j},\, u^1_{2j+1},\, u^1_{2j+2}\}.$$

Next, we will explain how to construct a good initial guess for the iterative solution by using extrapolation and quadratic interpolation.

From Eq. (4.1) we have

$$e^i_k=(u-u^i)(x_k)=A(x_k)h_i^\alpha+\mathcal{O}(h_i^\beta). \tag{4.4}$$

Here, if $i=0$, then $k=j,j+1$ for grid Z_0; and if $i=1$, then $k=2j,2j+1,2j+2$ for grid Z_1.

Extrapolation of the initial guess requires the use of the linear combination of the FE solutions u_0 and u_1 to approximate to the FE solution u_2, rather than to approximate the exact solution u of the problem.

Extrapolation at nodes: for a common node $x^0_k\in Z_0$, $(k=j,j+1)$, let the constant c satisfy the following equation:

$$cu^0_k+(1-c)u^1_{2k}=u^2_{4k}+\mathcal{O}(h_0^\beta). \tag{4.5}$$

When formula (4.5) is subtracted from the exact solution $u(x^0_k)$, we obtain

$$ce^0_k+(1-c)e^1_{2k}=e^2_{4k}+\mathcal{O}(h_0^\beta). \tag{4.6}$$

Substituting (4.4) into formula (4.6) yields

$$\left(c+\frac{1-c}{2^\alpha}\right)A(x^0_k)h_0^\alpha+\mathcal{O}(h_0^\beta)=\frac{1}{4^\alpha}A(x^0_k)h_0^\alpha+\mathcal{O}(h_0^\beta). \tag{4.7}$$

That is, $c=-1/2^\alpha$ is calculated. Then the extrapolation formula for a coarse grid node is

$$\tilde{u}^2_{4k}:=u^1_{2k}+\frac{u^1_{2k}-u^0_k}{2^\alpha}=u^2_{4k}+\mathcal{O}(h_0^\beta), \quad k=j,j+1. \tag{4.8}$$

Extrapolation at midpoints: it follows from the asymptotic error expansion (4.4) that we have

$$u^0_k-u^1_{2k}=\frac{2^\alpha-1}{2^\alpha}A(x^0_k)h_0^\alpha+\mathcal{O}(h_0^\beta), \quad k=j,j+1, \tag{4.9a}$$

$$u^1_{2j+1}-u^2_{4j+2}=\frac{2^\alpha-1}{4^\alpha}A(x^1_{2j+1})h_0^\alpha+\mathcal{O}(h_0^\beta). \tag{4.9b}$$

Using its Taylor expansion, we find

$$A(x_j^0)=A(x_{2j+1}^1)-A'(x_{2j+1}^1)h_{2j+1}^1+\mathcal{O}(h_0^2), \tag{4.10a}$$

$$A(x_{j+1}^0)=A(x_{2j+1}^1)+A'(x_{2j+1}^1)h_{2j+2}^1+\mathcal{O}(h_0^2). \tag{4.10b}$$

After eliminating $A'(x_{2j+1})$ by linearly combining formulae (4.10a) and (4.10b), then using formula (4.9a) we get

$$\begin{aligned}A(x_{2j+1}^1)=&\frac{h_{2j+1}^1A(x_{j+1}^0)+h_{2j+2}^1A(x_j^0)}{h_{j+1}^0}+\mathcal{O}(h_0^2)\\ =&\frac{2^\alpha}{(2^\alpha-1)h_0^\alpha}\frac{h_{2j+1}^1(u_{j+1}^0-u_{2j+2}^1)+h_{2j+2}^1(u_j^0-u_{2j}^1)}{h_{j+1}^0}+\mathcal{O}(h_0^2).\end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.9b) gives the high-precision extrapolation formula for the midpoint

$$\tilde{u}_{4j+2}^2:=u_{2j+1}^1+\frac{h_{2j+1}^1(u_{2j+2}^1-u_{j+1}^0)+h_{2j+2}^1(u_{2j}^1-u_j^0)}{2^\alpha h_{j+1}^0}=u_{4j+2}^2+\mathcal{O}(h_0^\beta). \tag{4.12}$$

Extrapolation at quartiles: by performing quadratic interpolation with the three obtained values, the high-precision initial values of the other two quartiles can be calculated. The relevant quadratic Lagrange polynomial is:

$$\begin{aligned}L_2(x)=&\tilde{u}_{4j}^2\frac{(x-x_{2j+1}^1)(x-x_{2j+2}^1)}{h_{2j+1}^1h_{j+1}^0}+\tilde{u}_{4j+2}^2\frac{(x-x_{2j}^1)(x-x_{2j+2}^1)}{-h_{2j+1}^1h_{2j+2}^1}\\ &+\tilde{u}_{4j+4}^2\frac{(x-x_{2j}^1)(x-x_{2j+1}^1)}{h_{2j+2}^1h_{j+1}^0}.\end{aligned} \tag{4.13}$$

By substituting $x=x_{4j+1}^2,x_{4j+3}^2$ into formula (4.13) and using (4.8) and (4.12), the extrapolation formula for the quartile is obtained.

For 2D problems, using a similar technique in [30] we can also obtain a good initial guess on finer mesh for the iterative solver. We omit the details here.

Remark 4.1. If uniform division is applied, that is, $h_{2j+1}^1=h_{2j+2}^1=h_{j+1}^0/2$, and the asymptotic error expansion (4.1) is replaced by

$$e_j^h=(u-u^h)(x_j)=A(x_j)h^2+\mathcal{O}(h_0^4), \tag{4.14}$$

then (4.8) and (4.12) coincide with the node extrapolation formula and the midpoint extrapolation formula in [14]:

$$\tilde{u}_{4k}^2:=u_{2k}^1+\frac{u_{2k}^1-u_k^0}{4}+\mathcal{O}(h_0^4),\quad k=j,j+1, \tag{4.15}$$

and

$$\tilde{u}^2_{4j+2} := u^1_{2j+1} + \frac{(u^1_{2j+2} - u^0_{j+1}) + (u^1_{2j} - u^0_j)}{8} = u^2_{4j+2} + \mathcal{O}(h_0^4). \tag{4.16}$$

And in this case, from Eq. (4.13) we have the following quartile extrapolation formulas:

$$\tilde{u}^2_{4j+1} := \frac{1}{16}\left[(9u^1_{2j} + 12u^1_{2j+1} - u^1_{2j+2}) - (3u^0_j + u^0_{j+1})\right] = u^2_{4j+1} + \mathcal{O}(h_0^3) \tag{4.17}$$

and

$$\tilde{u}^2_{4j+3} := \frac{1}{16}\left[(9u^1_{2j+2} + 12u^1_{2j+1} - u^1_{2j}) - (3u^0_{j+1} + u^0_j)\right] = u^2_{4j+3} + \mathcal{O}(h_0^3), \tag{4.18}$$

which mean that the quartile extrapolation formulas (4.17) and (4.18) provide a third-order approximation to the FE solution at x_{4j+1} and x_{4j+3}, see [15] for details.

Remark 4.2. It is also worth pointing out that, even though the extrapolation formulas (4.8) and (4.12) are needed to be derived from pointwise asymptotic error expansions, such as Eqs. (2.5) and (4.1), our EXCMG algorithm is still effective even for crack problems ($u \in H^{1.5-\epsilon}$), in which case using λ-graded meshes we have only the error estimates (3.4) and (3.5) in the L^2 norm, while extrapolation can also help us to obtain a higher-order approximation to the FE solution on the next finer grid in the L^2 norm, see numerical verification in Section 5: the last columns in Tables 1-7.

4.3 Steps of EXCMG algorithm

The key ingredients of the EXCMG method are extrapolation and quadratic interpolation, which are used to provide a better initial guess for the iterative solver on the next finer grid than one obtained by using linear interpolation in CMG. The algorithm steps are as follows:

Algorithm 1 EXCMG : $u_h \Leftarrow \text{EXCMG}(A_h, f_h, L, m_i)$.

1: $u_H \Leftarrow$ DSOLVE ($A_H u_H = f_H$)
2: $u_{H/2} \Leftarrow$ DSOLVE ($A_{H/2} u_{H/2} = f_{H/2}$)
3: $h = H/2$
4: **for** $i = 1$ to L **do**
5: $\quad h = h/2$
6: $\quad u_h = \text{EXP}(u_{2h}, u_{4h})$ $\quad \triangleright$ Calculate the initial guess for CG solver
7: $\quad u_h \Leftarrow \text{CG}(A_h, u_h, f_h, m_i)$ $\quad \triangleright$ Perform m_i CG iterations with the initial guess
8: **end for**

In Algorithm 1, a direct solver DSOLVE is used on the first two coarse grids since the sizes of the linear systems are not large. The procedure EXP(u_{2h}, u_{4h}) denotes a higher-order approximation to the FE solution u_h obtained by Richardson extrapolation and quadratic interpolation.

Generally, the number of CG iteration m_i is

$$m_i = \lfloor m_* \beta^{L-i} \rfloor, \tag{4.19}$$

where m_* refers to the number of iterations on the finest grid Z_L and is generally between 4 and 20: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. It is well known that multigrid complexity of the EXCMG method is obtained if and only if $\beta \leq 2^d$, where d is the dimension of the problem [5]. In our numerical experiments, we set $m_*=16$ and $\beta=4$.

It should be noted that once high-precision FE solutions on several fine grids are obtained, the extrapolation formulae (4.3) can be used to improve their precisions further [16]. The high-precision post-processing is one of the most significant advantages of this algorithm.

5 Numerical examples

In this section, some numerical experiments to verify the efficiency and effectiveness of our algorithm were computed.

5.1 A crack problem

Example 5.1. Consider a crack problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega = \{0<r<1,\ 0<\theta<2\pi\}, \\ u = g(r,\theta) & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a crack domain with maximum interior angle 2π, and the function $g(r,\theta)$ is determined from the exact solution

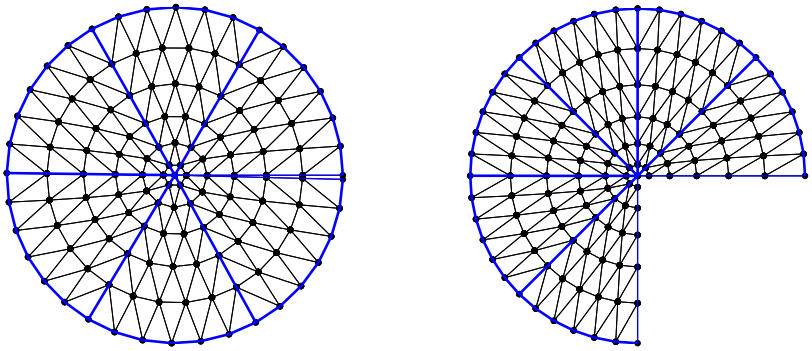
$$u = r^{1/2}\sin(\theta/2).$$

The exact solution u has a removable singularity at the origin and has only finite regularity in $H^{3/2-\epsilon}(\Omega)$ for any small $\epsilon > 0$.

We first divide the crack domain Ω into six sectors with central angle $\frac{\pi}{3}$, then a λ-graded grid can be used in each sector. By doing so, a piecewise λ-graded grid is obtained on the computational domain and all nodes lie on the arc S_j whose radius is $r_j = (jh)^\lambda$, $(j = 0,1,\cdots,N, h = 1/N)$, as shown in Fig. 3(a). We then use 7 embedded graded grids (including first two coarse grids where a direct linear solver is used) with the coarsest grid $N=15$, and the iteration number of the i-th level of the grid is set to be $m_i = 16\times 4^{5-i}$, which means that only 16 CG iterations are performed on the finest grid with $N = 960$ (the total number of grid points is $(3N+1)(N+1)$, nearly 3 million).

Let $E^k_{1/60}$ denote the difference between the FE solution $u_{1/60}$ and the k-th CG iterative solution $u^k_{1/60}$ with the initial guess obtained by extrapolating and interpolating the FE solutions $u_{1/15}$ and $u_{1/30}$,

$$E^k_{1/60} = u^k_{1/60} - u_{1/60} \to 0, \quad k \to \infty. \tag{5.2}$$



(a) A crack domain (b) An L-shaped domain

Figure 3: Piecewise λ-graded grid with $\lambda=1.5$ and $N=6$.

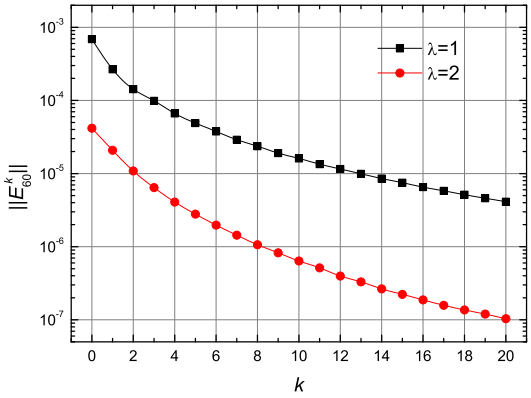


Figure 4: Convergence curve.

That is to say, $E_{1/60}^0$ represents the error between the extrapolated initial guess and the FE solution: it can be used to measure the effect of the extrapolation formula on the initial value; while $E_{1/60}^1$ refers to the error between the extrapolation value after performing a single CG iteration and the FE solution.

Fig. 4 shows the convergence curves of $\|E_{60}^k\|$, $(0\leq k\leq 20)$ versus the number of CG iterations k for $\lambda=1$ and $\lambda=2$. Here, $\|\cdot\|$ denotes the L^2 norm, which is defined by

$$\|u_h\|=\sqrt{\sum_{j=1}^{n_e}\iint_{e_j}u_h^2dxdy}, \tag{5.3}$$

where u_h is a grid function, n_e is the total number of the elements, and e_j denotes the j-th triangular element.

It is easily seen from Fig. 4 that when $\lambda=1$, the L^2 error between the initial guess and FE solution is already less than 10^{-3}; when $\lambda=2$, the initial error is further smaller,

Table 1: Errors and convergence orders of Example 5.1 with $\lambda=1$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	$2.51(-3)$		$3.69(-2)$		$1.90(-2)$		$6.88(-4)$	
120	$1.26(-3)$	0.99	$2.62(-2)$	0.49	$1.36(-2)$	0.49	$2.44(-4)$	1.50
240	$6.34(-4)$	1.00	$1.86(-2)$	0.50	$9.62(-3)$	0.49	$8.63(-5)$	1.50
480	$3.17(-4)$	1.00	$1.31(-2)$	0.50	$6.82(-3)$	0.50	$3.05(-5)$	1.50
960	$1.58(-4)$	1.01	$9.34(-3)$	0.49	$4.85(-3)$	0.49	$1.07(-5)$	1.51

Table 2: Errors and convergence orders of Example 5.1 with $\lambda=2$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	$1.79(-4)$		$6.39(-3)$		$2.92(-3)$		$4.17(-5)$	
120	$5.19(-5)$	1.78	$3.20(-3)$	1.00	$1.46(-3)$	1.00	$7.04(-6)$	2.57
240	$1.48(-5)$	1.81	$1.60(-3)$	1.00	$7.29(-4)$	1.00	$9.90(-7)$	2.83
480	$4.20(-6)$	1.82	$8.00(-4)$	1.00	$3.65(-4)$	1.00	$7.96(-8)$	3.64
960	$1.15(-6)$	1.86	$4.05(-4)$	0.98	$1.83(-4)$	0.99	$5.59(-9)$	3.83

achieving a precision of 10^{-4}. These two convergence curves indicate that the errors are significantly reduced with the growth of the number of CG iterations. As we can see that after only 3 iterations and the error has decreased one order of magnitude. Through 20 CG iterations, the precision of the iterative solution is improved by three orders of magnitude. So, as the number of iterations k increases, the required precision can be rapidly achieved. That's just the reason why our EXCMG algorithm is highly efficient for solving these problems.

We present the results of Example 5.1 with $\lambda=1,2,3$ and 4 in Table 1, Table 2, Table 3 and Table 4, respectively. Tables 1-4 list the L^2 error $\|u-u_h\|$ between the numerical solution u_h and the exact solution u, the energy-norm error $\|u-u_h\|_a$, the L^∞ error $\|u-u_h\|_\infty$, and the L^2 error $\|u_h^0-u_h\|$ between the initial guess u_h^0 and the numerical solution u_h.

As we can see from Table 1 that the initial guess is a 1.5-th order approximation to the FE solution in the sense of L^2 norm. And the numerical solution converges linearly (that is, with order 1) to the exact solution in the L^2 norm, while it converges only with order 0.5 in the L^∞ norm and energy norm, which validates the error estimate (2.3) since here $\gamma=0.5-\epsilon$ and $H^{1+\epsilon}(\Omega)$ can be continuously embedded into $L^\infty(\Omega)$ for two dimensional domain Ω.

From Table 2 we can see that the L^2 convergence order is near to 2 for $\lambda=2$, the FE solution converges linearly to the true solution in the L^∞ norm, and it achieves full linear convergence in the energy norm, which confirms the error estimate (3.5) stated in Theorem 3.1. Again, the initial guess is a higher-order approximation to the FE solution.

When $\lambda=3$ and $\lambda=4$, a full second order convergence rate in the L^2 norm can be seen from Tables 3-4, which confirms our theoretical results stated in Theorem 3.1 since here $\lambda>\alpha=2$. And the FE solution achieves 1.5-th order convergence in the L^∞ norm for $\lambda=3$,

Table 3: Errors and convergence orders of Example 5.1 with $\lambda=3$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	9.11(−5)		1.86(−3)		6.45(−4)		1.86(−05)	
120	2.31(−5)	1.98	6.66(−4)	1.48	2.28(−4)	1.50	2.37(−06)	2.97
240	5.82(−6)	1.99	2.37(−4)	1.49	8.08(−5)	1.50	3.00(−07)	2.98
480	1.46(−6)	2.00	8.40(−5)	1.50	2.86(−5)	1.50	3.77(−08)	2.99
960	3.70(−7)	1.98	2.97(−5)	1.50	1.01(−5)	1.50	4.78(−10)	6.30

Table 4: Errors and convergence orders of Example 5.1 with $\lambda=4$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	9.36(−5)		9.08(−4)		2.72(−4)		1.04(−05)	
120	2.35(−5)	2.00	2.51(−4)	1.86	7.02(−5)	1.96	8.49(−07)	3.62
240	5.87(−6)	2.00	6.82(−5)	1.88	1.78(−5)	1.98	6.54(−08)	3.70
480	1.47(−6)	2.00	1.83(−5)	1.90	4.49(−6)	1.99	4.03(−09)	4.02
960	3.69(−7)	2.00	4.89(−6)	1.90	1.13(−6)	1.99	4.06(−10)	3.31

while it reaches full second order convergence rate for $\lambda=4$. As we can see from Table 3 and Table 4 that the extrapolated initial guess is a third approximation to the FE solution for $\lambda=3$, and a nearly fourth order approximation to the FE solution for $\lambda=4$. It should be worth noting, that the FE solutions converge in the energy norm with an order equal to or larger than 1.5 for $\lambda=3$ and $\lambda=4$, which is a superconvergence result.

5.2 An L-shaped problem

Example 5.2. Consider the problem (5.1) over the L-shaped domain shown in Fig. 3(b). We select the function g so that the solution of the boundary value problem is

$$u=r^{2/3}\sin(2\theta/3), \tag{5.4}$$

which is the leading singularity associated with an L-shaped corner. The solution $u\in H^{5/3-\epsilon}(\Omega)$ for any small $\epsilon>0$.

Once again, we use 7 embedded grids with the coarsest grid $N=15$, and the iteration number $m_i=16\times 4^{5-i}$. We list the numerical results of Example 5.2 obtained by the EXCMG algorithm with $\lambda=1,2$ and 3 in Tables 5, 6 and 7, respectively.

From Table 5, it is easy to see that the initial guess is a 5/3-th order approximation to the FE solution in the sense of L^2 norm. And the numerical solution converges to the exact solution with order 4/3 in the L^2 norm, while it converges only with order 2/3 in the L^∞ norm and energy norm, which conforms to the error estimate (2.3) since here $\gamma=2/3-\epsilon$ and $H^{1+\epsilon}(\Omega)$ can be continuously embedded into $L^\infty(\Omega)$.

From Table 6 we can see that the L^2 convergence order is near to 2 for $\lambda=2$, and the FE solution converges to the true solution with order 4/3 in the L^∞ norm and the energy norm, which confirms the error estimate (3.5) stated in Theorem 3.1. Again, the

Table 5: Errors and convergence orders of Example 5.2 with $\lambda=1$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	$3.41(-4)$		$7.10(-3)$		$3.83(-3)$		$2.15(-4)$	
120	$1.39(-4)$	1.30	$4.48(-3)$	0.66	$2.42(-3)$	0.66	$6.79(-5)$	1.67
240	$5.57(-5)$	1.31	$2.82(-3)$	0.67	$1.53(-3)$	0.67	$2.14(-5)$	1.67
480	$2.24(-5)$	1.32	$1.78(-3)$	0.67	$9.61(-4)$	0.67	$6.74(-6)$	1.67
960	$9.05(-6)$	1.31	$1.13(-3)$	0.66	$6.10(-4)$	0.66	$2.11(-6)$	1.67

Table 6: Errors and convergence orders of Example 5.2 with $\lambda=2$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	$3.44(-5)$		$9.53(-4)$		$4.03(-4)$		$8.32(-6)$	
120	$8.80(-6)$	1.97	$3.79(-4)$	1.33	$1.60(-4)$	1.33	$1.25(-6)$	2.73
240	$2.23(-6)$	1.98	$1.51(-4)$	1.33	$6.35(-5)$	1.33	$1.92(-7)$	2.71
480	$5.69(-7)$	1.97	$5.99(-5)$	1.33	$2.52(-5)$	1.33	$2.64(-8)$	2.86
960	$1.52(-7)$	1.90	$2.38(-5)$	1.33	$1.00(-5)$	1.33	$1.89(-9)$	3.81

Table 7: Errors and convergence orders of Example 5.2 with $\lambda=3$.

N	$\|u-u_h\|$	order	$\|u-u_h\|_a$	order	$\|u-u_h\|_\infty$	order	$\|u_h^0-u_h\|$	order
60	$3.33(-5)$		$3.16(-4)$		$9.31(-5)$		$2.46(-06)$	
120	$8.35(-6)$	2.00	$8.68(-5)$	1.86	$2.38(-5)$	1.97	$1.90(-07)$	3.69
240	$2.09(-6)$	2.00	$2.35(-5)$	1.89	$6.01(-6)$	1.98	$1.29(-08)$	3.88
480	$5.25(-7)$	1.99	$6.30(-6)$	1.90	$1.51(-6)$	1.99	$1.23(-09)$	3.39
960	$1.34(-7)$	1.97	$1.69(-6)$	1.90	$3.89(-7)$	1.96	$6.28(-11)$	4.30

Table 8: Convergence orders of Example 5.1 and Example 5.2.

	θ_0	λ	$\|u-u_h\|$	$\|u-u_h\|_a$	$\|u-u_h\|_\infty$	$\|u_h^0-u_h\|$
Crack problem	2π	1	1	1/2	1/2	3/2
		2	≈ 2	1	1	≈ 3
		3	2	3/2	3/2	3
		4	2	≈ 2	2	≈ 4
L-shaped problem	$3\pi/2$	1	4/3	2/3	2/3	5/3
		2	2	4/3	4/3	≈ 3
		3	2	≈ 2	2	≈ 4

θ_0 denotes the maximum interior angle of the domains.

initial guess is a higher-order approximation to the FE solution, which greatly reduces the number of CG iterations required. For $\lambda=3$, we can see from Table 7 that the FE solution converges to the true solution with full order 2 in both the L^2 norm and the L^∞ norm. Further, the numerical solution u_h is a 1.9-th order approximation to the true solution in the energy norm, which is a superconvergence result. And the initial guess is a fourth order approximation to the FE solution. It should be noting that, on the finest mesh $N=960$, the maximum absolute error reaches 3.89×10^{-7}, which is much smaller

than those for $\lambda=1$ and 2. This indicates that the λ-graded mesh is very effective to solve such singular problems.

Table 8 summarizes the convergence orders of Example 5.1 and Example 5.2 for different λ in different norms. It shows that the rate of convergence of in the Crack problem in the energy norm is $3/2$ for $\lambda=3$, and nearly 2 for $\lambda=4$; the rate of convergence in the L-shaped problem in the energy norm is $4/3$ for $\lambda=2$, and nearly 2 for $\lambda=3$. These are superconvergence results of the FE method.

6 Conclusions

This work concerns an extrapolation cascadic multigrid method for solving the elliptic problems in domains with reentrant corners based on λ-graded grid. Based on the asymptotic error expansions of the FE approximation to the singular solution, a new extrapolation formula based on λ-graded grid is derived to obtain a good initial guess for the iterative solution on the next finer grid, which is a higher-order approximation to the FE solution. The obtained good initial guess greatly reduces the number of CG iterations required to achieve the expected accuracy.

Though the extrapolation formulas should be derived from the pointwise error estimates, our EXCMG algorithm is still effective even for crack problems ($u\in H^{1.5-\epsilon}$), in which case we have only the error estimate in the L^2 norm when using λ-graded meshes, while Richardson extrapolation can also help us to obtain a higher-order approximation to the FE solution in the L^2 norm on the next finer grid. Numerical experiments for a crack problem and an L-shaped problem are presented to verify the efficiency and the effectiveness of the EXCMG method for solving such singular elliptic problems. And the numerical results confirm the existing theoretical results.

Our method developed in this paper can be extended to solve three dimensional problems in spherical coordinates. We are currently investigating these extensions.

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