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Asymptotic Exponential Arbitrage in a Liu-Tang 3-Factor Model of Commodity Futures

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Abstract. From a 3-factor model of storable commodities discussed by Liu and Tang (2010), we consider a cash market model such as futures exchange with a single futures contract on one such commodities and a money market account. After verifying that this model is arbitrage-free and incomplete in any finite time horizon or delivery date, we show that there still exists a possibility to generate exponentially growth risk-less profit in long term; a form of asymptotic arbitrage conjectured by Föllmer and Schachermayer (2008) and first solved by Mbele Bidima and Rásonyi (2012) in financial security models. And we find that works in this paper generalize our recent works in Tadesse Welemical *et al.* (2019) on Schwartz's one-factor model of commodity futures.

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1 Introduction

A key challenge for market makers and policymakers is to avoid losses while guaranteeing some (risk-less) profits in long-term economic trading. Such profits were first studied under a concept named "asymptotic arbitrage" by Kabanov and Kramkov [13] in their pioneering work. The concept gained some incredible development over the past decade after the contributing and inspiring works of Föllmer and Schachermayer [8]. It is an emerging theory in modern Mathematical Finance where authors are analyzing existence of arbitrage opportunities (risk-less profits) in long-term trad-

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ing, i.e. when the trading time horizon T tends to infinity, in general (or typical) financial models. This concept is discussed depending on whether classical arbitrage in any finite time horizon appear or not in those models as guaranteed by the First Fundamental Theorem of Asset Pricing (stated in [4, 12]). However, the literature in this subject is still narrow and shows that the analysis of asymptotic arbitrage has been carried out essentially by a limited number of authors in models of financial securities only, see for instance [3, 6, 11, 14, 16, 17].

A couple of years ago [20], we were the first to conduct analysis of asymptotic arbitrage in models of storable commodities, especially in the Schwartz's one-factor model of storable commodity futures. In that article, we carefully started with highlighting the difference between standard securities markets and (storable) commodities market. Indeed, unlike financial securities (such as stocks, bonds, etc.), storable commodities (like oil, gold, coffee, etc.) are characterized by an unavoidable cost of carrying a physical good, also known as the convenience yield. And they are traded in two interrelated markets: the storage (inventory) market where physical commodities are assumed traded at prices called spot prices, and the cash market (typically a futures exchange) where futures contracts on those commodities with maturity time (delivery date) T are traded at time $t \leq T$ with prices known as futures prices.

In Schwartz's one factor model of storable commodities [18], the convenience yield is assumed constant, and this was crucial in our analysis of asymptotic arbitrage in [20]. But in a number of other classical models of (storable) commodities such as [10], the convenience yield is not constant and is modeled as an Ornstein-Uhlenbeck (OU) process, which can take negative values. But Liu and Tang [15, Lemma 1, Theorem 1] proved that under short-selling prohibiting, the negativity of the convenience yield is equivalent to existence of arbitrage opportunities in the storage market in any finite trading time horizon. To overcome such a limitation, the authors of [15] modeled the convenience yield using a Cox-Ingersoll-Ross (CIR) process. As a result, they developed a so-called semi-affine 3-factor model (with factors being the spot price of a commodity, the convenience yield and the short interest rate) under an equivalent martingale measure (EMM) $\mathbb{Q}$ existing in the cash market. Since the CIR process assumes the non-negativity constraint for the underlying (the convenience yield in this case), then this guarantees the requirement of no-arbitrage opportunity in their such storage market model in any finite time horizon.

In this paper, we consider in the section below the so-called Liu-Tang 3-factor commodity futures model (Definition 2.1) which we built on the setup of the Liu-Tang 3-factor model for a storable commodity that follows. We verify in Section 3 that this constructed 3-factor futures model is arbitrage-free and incomplete in any finite time horizon (future delivery date) $T > 0$. Next in Section 4, which is the main part of our article, after recalling the concept asymptotic exponential arbitrage with geometrically decaying failure probability which discussed in [20] for a Schwartz's one factor commodity futures model, we prove in Theorem 4.1 existence of such trading opportunities in the Liu-Tang 3-factor commodity futures model. And we end the paper with a conclusion and some perspectives in Section 5.

2 The Liu-Tang 3-factor commodity futures model

Consider the Liu-Tang 3-factor model where the three factors are respectively a spot price process S_t for a single storable commodity in a storage market, the short rate process r_t of trading such a commodity and a convenience yield process δ_t, all assumed continuous, adapted and verifying sufficient stochastic integrability requirements on the same filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Next, beside this storage market, we assume there is a cash market (a futures exchange) with a single futures contract on the above storable commodity whose prices process is denoted $F_t \equiv F(t,T) := F(S_t, \delta_t, r_t, t, T)$ for any future delivery date (or time horizon) $T > 0$. We also assume there is an accompanying money market account serving as a numéraire (typically a risk-free bond) with deterministic price B_t at any time $t \geq 0$.

Definition 2.1 (Liu-Tang 3-Factor Commodity Futures Model). *i) In our present paper, we call "Liu-Tang 3-factor commodity futures model" any market model with the two prices processes F_t (of a futures contract on the storable commodity in the Liu-Tang 3-factor model) and B_t of the accompanying numéraire.*

ii) According to [15]*, a trading portfolio in such a commodity futures model is a pair of $\mathbb{R}$-valued and $\mathbb{F}$-adapted processes $\pi_t := (\theta_t, \varphi_t)$ representing respectively the investor's position in the risk-free bond and in the futures contract.*

iii) The value process of such a trading portfolio (or wealth of the investor) denoted V_t^π is defined by

$$V_t^\pi := \theta_t B_t + \varphi_t F_t, \quad \forall\, t \in [0,T]. \tag{2.1}$$

iv) Any such trading opportunity is said to be self-financing if it is a pair of predictable processes and verifies

$$dV_t^\pi := \theta_t dB_t + \varphi_t dF_t. \tag{2.2}$$

Assumption 2.1. *For computational simplicity, we assume that the risk-free interest rate on the bond is zero, i.e. the bond price is $B_t := 1$ for all $t \in [0,T]$.*

Remark 2.1. 1) This assumption implies that the discounted futures prices process $\tilde{F}_t := F_t/B_t$ is just F_t and the discounted value $\tilde{V}_t^\pi := V_t^\pi/B_t$ of any portfolio π_t is again V_t^π.

2) Similar to the restriction assumed in [1, Section 2.2], the assumption above hints that we may restrict the definition of self-financing portfolios to the part φ_t of investment in the futures contract only. Hence, denoting the value process now $V_t^\pi \equiv V_t^\varphi$, the self-financing relation above becomes

$$dV_t^\varphi = \varphi_t dF_t, \quad \text{i.e.} \quad V_t^\varphi = V_0^\varphi + \int_0^t \varphi_s dF_s, \quad \forall\, t \in [0,T]. \tag{2.3}$$

Definition 2.2 (Admissible Strategy). *A self-financing trading portfolio φ_t is said to be admissible, if there is a constant $a \geq 0$ such that its corresponding value process V_t^{φ} verifies the constraint $V_t^{\varphi} \geq -a$, $\mathbb{P}-a.s.$ for all time $t \in [0, T]$.*

Let $\mathcal{H}$ be the set of admissible self-financing strategies in this model. For any time $t > 0$ (in particular for time horizon $T > 0$), denote the set of value processes for all admissible self-financing strategies φ_s up to time t with $V_0^{\varphi} = 0$ as $\mathcal{K}_t$, i.e.

$$\mathcal{K}_t := \left\{ \int_0^t \varphi_s dF_s : \varphi_s \in \mathcal{H} \text{ with } V_0^{\varphi} = 0 \right\}. \tag{2.4}$$

Under the no-arbitrage assumption, equivalent to the existence of an equivalent (local) martingale measure (EMM) $\mathbb{Q}$ in the cash market (as provided by the fundamental theorem of asset pricing in [4]), it is well known (see [9, p. 31]) that the (discounted) futures price F_t at any time $t \leq T$ is given by the relation

$$F_t = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] =: \mathbb{E}_t^{\mathbb{Q}}[S_T], \tag{2.5}$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation taken under any such (local) EMM $\mathbb{Q}$.

Next, under any such EMM $\mathbb{Q}$, Liu and Tang [15] assumed that the three factors S_t, r_t and δ_t are governed by the following dynamics:

$$\begin{cases} dr_t = \kappa_1(\mu_1 - r_t)dt + \sigma_1\sqrt{r_t}dW_t^1, & \text{(2.6a)} \\ d\delta_t = \kappa_2(\mu_2 - \delta_t)dt + \sigma_2\sqrt{\delta_t}dW_t^2, & \text{(2.6b)} \\ dS_t = [(r_t - \delta_t)S_t + w]dt + S_t\sigma_{Sr}\sqrt{r_t}dW_t^1 + S_t\sigma_{S\delta}\sqrt{\delta_t}dW_t^2 & \\ \qquad +S_t\sqrt{v_0 + v_{S\delta}\delta_t + v_{sr}r_t}dW_t^3, & \text{(2.6c)} \end{cases}$$

where W_t^1, W_t^2 and W_t^3 are each other independent Brownian motions under $\mathbb{Q}$, $\mu_1, \kappa_1 > 0$ are respectively the long run mean and speed of mean reversion of the interest rate with constant volatility $\sigma_1 > 0$, $\mu_2, \kappa_2 > 0$ are respectively the long run mean and the rate of mean reversion for the convenience yield with constant volatility $\sigma_2 > 0$, v_0, $v_{S\delta}$ and v_{Sr} are $\mathbb{R}$-valued non negative constants and w is the storage cost.

According to [15], the equivalent martingale measure $\mathbb{Q}$ and the probability measure $\mathbb{P}$ are related as

$$d\begin{bmatrix} W_t^1 \\ W_t^2 \\ W_t^3 \end{bmatrix} = \begin{bmatrix} \eta_1\sqrt{r_t} \\ \eta_2\sqrt{\delta_t} \\ \eta_3\sqrt{v_0 + v_{S\delta}\delta_t + v_{Sr}r_t} \end{bmatrix} dt + d\begin{bmatrix} Z_t^1 \\ Z_t^2 \\ Z_t^3 \end{bmatrix}, \tag{2.7}$$

where η_1, η_2, η_3 are real constants and Z_t^1, Z_t^2, Z_t^3 are standard Brownian motions under the physical measure $\mathbb{P}$ such that

$$dZ_t^1 dZ_t^2 = dZ_t^1 dZ_t^3 = dZ_t^2 dZ_t^3 = 0,$$

and the vector

$$\psi_t := \big(\psi_1(t), \psi_2(t), \psi_3(t)\big) := \Big(\eta_1\sqrt{r_t}, \eta_2\sqrt{\delta_t}, \eta_3\sqrt{v_0 + v_{S\delta}\delta_t + v_{Sr}r_t}\Big) \tag{2.8}$$

is the market price of risk for the Liu-Tang 3-factor model, i.e. $\psi_i(t)$ for $i = 1, 2, 3$, are respectively market prices of risk of the factors r_t, δ_t and S_t corresponding to respective standard Brownian motions W_t^i for $i = 1, 2, 3$.

Remark 2.2. Since the convenience yield $\delta_t \geq 0$ by construction of the CIR process in the Eq. (2.6b), we may even assume in this paper that $\delta_t > 0$ for all time $t \geq 0$, then as pointed out from the introduction, [15, Lemma 1, Theorem 1] imply that the Liu-Tang 3-factor commodity model (all equations in (2.6)) does not allow short-selling and is arbitrage-free for any finite time horizon $T > 0$.

3 Absence of finite horizon arbitrage and model incompleteness

Recall the following classical definitions in Mathematical Finance.

Definition 3.1 (Finite Horizon Arbitrage). *i) For any finite time horizon $T > 0$, a (self-financing) portfolio φ_t is an arbitrage opportunity in this futures market model if its corresponding value process V_t^{φ}, $t \in [0, T]$, satisfies the following conditions:*

$$V_0^{\varphi} = 0, \quad V_T^{\varphi} \geq 0, \quad \mathbb{P} - a.s. \quad with \quad \mathbb{P}\big(V_T^{\varphi} > 0\big) > 0. \tag{3.1}$$

ii) The model is said to be arbitrage-free for any finite time horizon $T > 0$ if there is no arbitrage opportunity in the sense above.

Definition 3.2 (Model Completeness). *A market model is said to be complete if every contingent claim is attainable i.e. there is a self-financing strategy whose final value equals the payoff of the claim, otherwise it is said to be incomplete.*

In the specific case of their model, Liu and Tang [15] derived the following explicit formula for the futures prices.

Proposition 3.1 ([15, Proposition 4]). *Assume (for computational simplicity) that the storage cost $w = 0$. Then the (discounted) futures price process F_t for the Liu-Tang 3-factor commodity model is given as*

$$F_t = S_t \exp\big(A(t,T) + B(t,T)r_t + C(t,T)\delta_t\big), \tag{3.2}$$

where

$$A(t,T) = \frac{2\kappa_2\mu_2}{\sigma_2^2}\left[(T-t)b_1 + \ln\left(1 - \frac{b_1}{b_2}\right) - \ln\left(1 - \frac{b_1}{b_2}\exp\big((b_1 - b_2)(T-t)\big)\right)\right]$$

$$+\frac{2\kappa_1\mu_1}{\sigma_1^2}\left[(T-t)b_3+\ln\left(1-\frac{b_3}{b_4}\right)-\ln\left(1-\frac{b_3}{b_4}\exp\big((b_3-b_4)(T-t)\big)\right)\right],$$

$$B(t,T)=\frac{2b_1}{\sigma_2^2}\left[\frac{1-\exp\big((b_1-b_2)(T-t)\big)}{1-b_1\exp\big((b_1-b_2)(T-t)\big)/b_2}\right],$$

$$C(t,T)=\frac{2b_3}{\sigma_1^2}\left[\frac{1-\exp\big((b_3-b_4)(T-t)\big)}{1-b_3\exp\big((b_3-b_4)(T-t)\big)/b_4}\right]$$

with

$$\begin{aligned}
b_1&=-\frac{1}{2}(\sigma_2\sigma_{S\delta}-\kappa_2)-\frac{1}{2}\sqrt{(\sigma_2\sigma_{S\delta}-\kappa_2)^2+2\sigma_2^2},\\
b_2&=-\frac{1}{2}(\sigma_2\sigma_{S\delta}-\kappa_2)+\frac{1}{2}\sqrt{(\sigma_2\sigma_{S\delta}-\kappa_2)^2+2\sigma_2^2},\\
b_3&=-\frac{1}{2}(\sigma_1\sigma_{Sr}-\kappa_1)-\frac{1}{2}\sqrt{(\sigma_1\sigma_{Sr}-\kappa_1)^2-2\sigma_1^2},\\
b_4&=-\frac{1}{2}(\sigma_1\sigma_{Sr}-\kappa_1)+\frac{1}{2}\sqrt{(\sigma_1\sigma_{Sr}-\kappa_1)^2-2\sigma_1^2}.
\end{aligned}$$

From this we state the following useful result.

Proposition 3.2. *1) The (discounted) futures prices process F_t in* (3.2) *obeys the dynamics*

$$dF_t = D_tF_t\cdot dW_t \tag{3.3}$$

under any EMM $\mathbb{Q}$ in the cash market, where

$$\begin{aligned}
D_t &:= \left((\sigma_{Sr}+\sigma_1B_t)\sqrt{r_t},(\sigma_{S\delta}+\sigma_2C_t)\sqrt{\delta_t},\sqrt{v_0+v_{Sr}r_t+v_{S\delta}\delta_t}\right),\\
W_t &:= \big(W_t^1,W_t^2,W_t^3\big).
\end{aligned}$$

2) Assuming $v_0=v_{Sr}=v_{S\delta}=0$ (again for calculation simplicity), then under the physical measure $\mathbb{P}$ the (discounted) futures price in (3.3) *is expressed as*

$$dF_t = D_tF_t\cdot\begin{bmatrix}\psi_1dt+dZ_t^1\\ \psi_2dt+dZ_t^2\\ 0+dZ_t^3\end{bmatrix}, \tag{3.4}$$

where the market price of risk for the futures contract is the same as the market price of risk

$$\psi_t := \big(\psi_1(t),\psi_2(t),\psi_3(t)\big)=\big(\eta_1\sqrt{r_t},\eta_2\sqrt{\delta_t},0\big)$$

of the original Liu-Tang three factors in (2.8) *and depends on the two factors r_t and δ_t only.*

3) Hence, according to (2.3) *a trading strategy φ_t in the Liu-Tang commodity futures model is self-financing if and only if its (discounted) value process satisfies*

$$dV_t^{\varphi} = \varphi_tD_tF_t\cdot\begin{bmatrix}\psi_1dt+dZ_t^1\\ \psi_2dt+dZ_t^2\\ 0+dZ_t^3\end{bmatrix}. \tag{3.5}$$

Proof. It is an application of Itô lemma on F_t. □

Next, what Liu and Tang [15] missed to highlight and which is crucial to our analysis of asymptotic arbitrage in the next subsection is that the equivalent martingale measure $\mathbb{Q}$ in the cash market is no longer unique in the case of their 3-factor model. Indeed.

Proposition 3.3 (No-Arbitrage and Model Incompleteness)**.** *The cash market model (the Liu-Tang commodity futures model) is arbitrage-free and incomplete in any finite time horizon $T > 0$.*

Proof. The dynamics in (3.4) shows that the (discounted) futures contract, which is the only risky asset in that cash market, is driven by three independent random sources i.e. by the three Brownian motions Z_t^1, Z_t^2 and Z_t^3. This implies by [2, Meta-Theorem 8.3.1] that the Liu-Tang commodity futures model is incomplete and arbitrage-free for any finite delivery date (time horizon) $T > 0$. □

4 Existence of asymptotic exponential arbitrage

Recall first that the futures prices of the underlying commodity is $F_t \equiv F(t, T)$ for all times $t \in [0, T]$, where $T > 0$ is a fixed time horizon, and that for any admissible self-financing portfolio φ_t in $\mathcal{H}$, the investors' wealth in the futures contract in (3.2) is

$$V_t^\varphi = V_0^\varphi + \int_0^t \varphi_s dF_s$$

at any time $t \in [0, T]$.

Notice that unlike in financial security models, since the futures price depends on two time parameters t and T, to discuss the asymptotic behavior of the wealth process V_t^φ i.e. from long-term trading in the futures contract when the delivery date T becomes larger and larger, it is enough to discuss it when the running time t is getting larger and larger since $t \leq T$.

Next, Mbele Bidima and Rásonyi [16,17] gave a better mathematical formulation of the following form of asymptotic arbitrage inspired by Föllmer and Schachermayer [8] and which we adapt here in our present modeling setting of commodity futures.

Definition 4.1 ([16, Definition 1.1])**.** *We say that the futures prices process F_t generates asymptotic exponential arbitrage (AEA) with geometrically decaying failure probability (GDFP), if there are constants $C, \lambda_1, \lambda_2 > 0$ and a trading strategy $\varphi_t \in \mathcal{H}$ such that the value process V_t^φ satisfies the following conditions:*

(i) $V_t^\varphi \geq -e^{-\lambda_1 t}$, $\mathbb{P} - a.s.$,

(ii) $\mathbb{P}[V_t^\varphi \geq e^{\lambda_1 t}] \geq 1 - Ce^{-\lambda_2 t}$ *for large enough time $t > 0$, or equivalently*

(ii)' $\mathbb{P}[V_t^\varphi < e^{\lambda_1 t}] \leq Ce^{-\lambda_2 t}$ *for large enough time $t > 0$.*

This definition presents a better economic interpretation: It says that the maximal loss of the investor's wealth at any time t is exponentially bounded in (i) by $e^{-\lambda_1 t}$ and (ii) means that, even from zero initial capital, an investor may generate a profit that grows exponentially fast in time with probability converging to 1 exponentially (geometrically) fast. And (ii)′ states that failing to achieve such a growth profit can be controlled in time by a probability converging to 0 exponentially fast.

Our main goal as announced in this paper is to find trading opportunities φ_t generating that form of asymptotic arbitrage in the Liu-Tang 3-factor commodity futures model (3.4) despite classical arbitrage opportunities do not exist in any finite time horizon as highlighted in Proposition 3.3. For this purpose, note that Proposition 3.3 entitles existence of several equivalent (local) martingale measures in our constructed Liu-Tang 3-factor commodity futures model i.e. probability measures $\mathbb{Q} \sim \mathbb{P}$ (i.e. equivalent to $\mathbb{P}$) and under which the (discounted) futures prices process is a (local) martingale. We denote $\mathcal{M}_t^e$ the set of such measures $\mathbb{Q}$ in the model up to any future trading date $t > 0$ (in particular any delivery date $T > 0$).

As similarly argued in [6, Section 1] and [11, Section 2.2], we may consider this class $\mathcal{M}_t^e$ defined in terms of the Radon-Nikodym densities

$$L_t := \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\int_0^t \psi_s \cdot dW_s - \frac{1}{2}\int_0^t \|\psi_s\|^2 ds\right\} \tag{4.1}$$

for any time $t > 0$, where ψ_t is the market price of risk vector for the futures prices given in (3.4) (or for the 3-factor model as given in (2.8)),

$$dW_t = \left(dW_t^1, dW_t^2, dW_t^3\right),$$

and $\|\psi_t\|$ is the Euclidean norm of ψ_t given by

$$\|\psi_t\| := \sqrt{\psi_1^2(t) + \psi_2^2(t) + \psi_3^2(t)}.$$

We state below the main theorem of this paper.

Theorem 4.1. *There exists an admissible self-financing strategy φ_t that allows the futures prices process F_t to generate asymptotic exponential arbitrage with geometrically decaying failure probability in the Liu-Tang 3-factor commodity futures model.*

The proof of this theorem is based on a technical mid-way result to asymptotic arbitrage stated in a general semi-martingale setting in [8] and on the following preliminary set of results which we prove applying those on Large Deviations Theory from [5,7]. First, we state and prove the key lemma below.

Lemma 4.1. *Let Y_t and X_t be the stochastic processes defined for all time $t > 0$ by*

$$Y_t := \frac{1}{t}\left(\gamma_1\delta_t + \gamma_2\int_0^t \delta_s ds\right), \quad X_t := \frac{1}{t}\left(\alpha_1 r_t + \alpha_2\int_0^t r_s ds\right), \tag{4.2}$$

where $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ are $\mathbb{R}$-valued constants, but assuming $\gamma_2, \alpha_2 > 0$. Then Y_t and X_t satisfy the large deviations principle in $\mathbb{R}$ respectively with good rate functions I_2 and I_1 defined on $\mathbb{R}$ as follows, for $x \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$,

$$I_2(x) = \frac{(\kappa_2 x - \kappa_2 \mu_2 \gamma_2)^2}{2\sigma_2^2 \gamma_2 |x|}, \quad I_1(x) = \frac{(\kappa_1 x - \kappa_1 \mu_1 \alpha_2)^2}{2\sigma_1^2 \alpha_2 |x|}, \quad I_2(0) = \infty = I_1(0). \tag{4.3}$$

Proof. Solving the second stochastic differential equation in (2.6), we get

$$\delta_t = \delta_0 e^{-\kappa_2 t} + \mu_2(1 - e^{-\kappa_2 t}) + \sigma_2 \int_0^t e^{-\kappa_2(t-s)} \sqrt{\delta_s} dW_s^2, \quad \delta_0 > 0. \tag{4.4}$$

For any $t > 0$, let $M_t(y)$ be the moment generating function of Y_t at some $y \in \mathbb{R}$, then analogously to the computations made in the proof of [7, Lemma 4.1] and using the definition of the confluent function of the first kind $R(c, d, z)$, for complex numbers with $Re(c), Re(d) > 0$,

$$R(c, d, z) := \frac{\Gamma(b)}{\Gamma(d - c)\Gamma(c)} \int_0^1 e^{zt} t^{c-1} (1 - t)^{d-c-1} dt \tag{4.5}$$

with Γ being the gamma function defined on $\mathbb{R}$ as

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R},$$

we get

$$\begin{aligned}
M_t(y) &= \mathbb{E}\left[\exp\left(\frac{\gamma_1 \delta_t y + \gamma_2 y}{t} \int_0^t \delta_s ds\right)\right] \\
&= \frac{\Gamma\big(a + (b+1)/2\big)}{\Gamma(b+1)} \exp\left[\frac{\kappa_2}{\sigma_2^2}(\kappa_2 \mu_2 t + \delta_0) - \frac{\beta \delta_0}{\sigma_2^2} \frac{\cosh(\beta t/2)}{\sinh(\beta t/2)}\right] \\
&\quad \times \left(\frac{\beta \delta_0}{\sigma_2^2 \sinh(\beta t/2)}\right)^{\frac{1}{2}(b+1) - a} \times \left(\frac{\kappa_2}{\beta} \sinh\left(\frac{\beta t}{2}\right) + \cosh\left(\frac{\beta t}{2}\right)\right)^{-(a + \frac{1}{2}(b+1))} \\
&\quad \times R\left(a + \frac{1}{2}(b+1), b+1, \frac{\beta^2 \delta_0}{\sigma_2^2 \sinh(\beta t/2) \left[\kappa_2 \sinh(\beta t/2) + \cosh(\beta t/2)\right]}\right),
\end{aligned} \tag{4.6}$$

where

$$a := \frac{\kappa_2 \mu_2}{\sigma_2^2}, \quad b := \frac{2\big(\kappa_2 \mu_2 - \sigma_2^2/2\big)}{\sigma_2^2} = 2a - 1, \quad \beta := \sqrt{\kappa_2^2 - \frac{2\sigma_2^2 \gamma_2 y}{t}}.$$

Next for suitable $y \in \mathbb{R}$, define $\Lambda(y)$ as

$$\Lambda(y) := \lim_{t \to \infty} \frac{1}{t} \ln M_t(ty). \tag{4.7}$$

Since

$$\lim_{t\to\infty}\frac{1}{t}\ln\frac{\Gamma\big(a+(b+1)/2\big)}{\Gamma(b+1)}=0,$$
$$\lim_{t\to\infty}\frac{1}{t}\ln R\left(a+\frac{1}{2}(b+1),b+1,\frac{\beta^2\delta_0}{\sigma_2^2\sinh(\beta t/2)\,[\kappa_2\sinh(\beta t/2)+\cosh(\beta t/2)]}\right)=0,$$

then similar to the derivation in the proof of [7, Lemma 4.2], we have

$$\begin{aligned}
\Lambda(y) &= \lim_{t\to\infty}\frac{1}{t}\ln M_t(ty)\\
&= \lim_{t\to\infty}\frac{1}{t}\left[\frac{\kappa_2}{\sigma_2^2}(\kappa_2\mu_2 t+\delta_0)-\frac{\beta\delta_0}{\sigma_2^2}\frac{\cosh(\beta t/2)}{\sinh(\beta t/2)}\right]\\
&\quad+\lim_{t\to\infty}\frac{1}{t}\left(\frac{b+1}{2}-a\right)\ln\left(\frac{\beta\delta_0}{\sigma_2^2\sinh(\beta t/2)}\right)\\
&\quad-\lim_{t\to\infty}\frac{1}{t}\left(a+\frac{b+1}{2}\right)\ln\left(\frac{\kappa_2}{\beta}\sinh\left(\frac{\beta t}{2}\right)+\cosh\left(\frac{\beta t}{2}\right)\right)\\
&= \frac{\kappa_2^2\mu_2}{\sigma_2^2}-\frac{b\beta}{2}-\frac{\beta}{2}\\
&= \frac{\kappa_2^2\mu_2}{\sigma_2^2}-\frac{2}{2\sigma_2^2}\left(\kappa_2\mu_2-\frac{\sigma_2^2}{2}\right)\sqrt{\kappa_2^2-2\sigma_2^2\gamma_2 y}-\frac{1}{2}\sqrt{\kappa_2^2-2\sigma_2^2\gamma_2 y}\\
&= \frac{\kappa_2^2\mu_2}{\sigma_2^2}-\frac{1}{2}\sqrt{\kappa_2^2-2\sigma_2^2\gamma_2 y}\left(\frac{2\kappa_2\mu_2-\sigma_2^2}{\sigma_2^2}+1\right)\\
&= \frac{\kappa_2^2\mu_2}{\sigma_2^2}-\frac{1}{2}\sqrt{\kappa_2^2-2\sigma_2^2\gamma_2 y}\left(\frac{2\kappa_2\mu_2}{\sigma_2^2}\right)\\
&= \frac{\kappa_2\mu_2}{\sigma_2^2}\left(\kappa_2-\sqrt{\kappa_2^2-2\sigma_2^2\gamma_2 y}\right).
\end{aligned}\tag{4.8}$$

Recalling we assumed $\gamma_2>0$, then $\Lambda(y)$ so defined is a real number only if $y\in(-\infty,\kappa_2^2/2\sigma_2^2\gamma_2]$. Hence, we may extend the definition of $\Lambda(y)$, for any $y\in\mathbb{R}$, as an extended real number by setting

$$\Lambda(y)=\begin{cases}\dfrac{\kappa_2\mu_2}{\sigma_2^2}\left(\kappa_2-\sqrt{\kappa_2^2-2\sigma_2^2\gamma_2 y}\right), & \text{if}\quad y\le\dfrac{\kappa_2^2}{2\sigma_2^2\gamma_2},\\[2ex] \infty, & \text{if}\quad y>\dfrac{\kappa_2^2}{2\sigma_2^2\gamma_2}.\end{cases}\tag{4.9}$$

Obviously the function Λ is convex and differentiable in the interior of its effective domain

$$\mathcal{D}_\Lambda:=\{y\in\mathbb{R}:\Lambda(y)<\infty\}=\left(-\infty,\frac{\kappa_2^2}{2\sigma_2^2\gamma_2}\right],$$

which contains the origin 0. It is easy to check that it verifies the remaining conditions of Gärtner-Ellis' theorem (see [5, Theorem 2.3.6] and its such conditions [5, Assumption 2.3.2 and Definition 2.3.5]), which implies that the process Y_t satisfies the large deviations principle (LDP) with good rate function $I_2 := \Lambda^*$, the convex conjugate of Λ, i.e.

$$-\inf_{x \in U} I_2(x) \le \liminf_{t\to\infty} \frac{1}{t} \ln \mathbb{P}[Y_t \in U] \le \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}[Y_t \in G] \le -\inf_{x\in G} I_2(x) \quad (4.10)$$

for any open subset U of $\mathbb{R}^*$ and closed subset G of $\mathbb{R}^*$.

Next, by definition, for all $x \in \mathbb{R}$, we have:

1. If $x = 0$, then $\Lambda^*(0)$ is given by

$$\Lambda^*(0) = \sup_{y \in \mathcal{D}_\Lambda} \{0 \times y - \Lambda(y)\} = \infty \quad \text{obviously.} \quad (4.11)$$

2. If $x \neq 0$, then

$$\Lambda^*(x) = \sup_{y \in \mathcal{D}_\Lambda} \{xy - \Lambda(y)\}. \quad (4.12)$$

The supremum here in (4.12) is obtained when $\Lambda'(y) = x$. But since

$$\Lambda'(y) = \frac{\kappa_2 \mu_2 \gamma_2}{\sqrt{\kappa_2^2 - 2\sigma_2^2 \gamma_2 y}},$$

then this supremum is obtained when

$$y = \frac{\kappa_2^2 x^2 - (\kappa_2\mu_2)^2 \gamma_2^2}{2\sigma_2^2 \gamma_2 x^2} = \frac{\kappa_2^2}{2\sigma_2^2\gamma_2} - \frac{(\kappa_2\mu_2)^2\gamma_2}{2\sigma_2^2 x^2} \in \mathcal{D}_\Lambda. \quad (4.13)$$

Therefore,

$$\Lambda^*(x) = \frac{(\kappa_2 x - \kappa_2\mu_2\gamma_2)^2}{2\sigma_2^2\gamma_2|x|}. \quad (4.14)$$

This allows to obtain the exact formula for the rate function $I_2 := \Lambda^*$, taking extended real numbers values as stated in (4.3).

Using similar arguments, the process X_t satisfies the large deviations principle with good rate function I_1 given in (4.3), ending the proof of the lemma. □

Next, since W_t^1 and W_t^2 are independent Brownian motions, then for all real constants $\lambda_1, \lambda_2 > 0$, and for any time $t > 0$, $\{L_t^1 \ge e^{-\lambda_1 t}\}$ and $\{L_t^2 \ge e^{-\lambda_2 t}\}$ are independent events, where

$$L_t^1 := \exp\left(-\int_0^t \psi_1(s)dW_s^1 - \frac{1}{2}\int_0^t \psi_1^2(s)ds\right), \quad (4.15)$$

$$L_t^2 := \exp\left(-\int_0^t \psi_2(s)dW_s^2 - \frac{1}{2}\int_0^t \psi_2^2(s)ds\right), \quad (4.16)$$

and $\psi_1(t), \psi_2(t)$ are respectively market prices of risk for the interest rate r_t and the convenience yield δ_t.

Then we state and prove the following result.

Proposition 4.1. *Let the constants* $\lambda_1, \lambda_2 > 0$ *be such that*

$$\lambda_1 \leq \frac{\mu_1 \eta_1^2}{2}, \quad \lambda_2 \leq \frac{\mu_2 \eta_2^2}{2}. \tag{4.17}$$

Define for all time $t > 0$ *the event*

$$A_t := \left\{ L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}} \right\} \cap \left\{ L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}} \right\}. \tag{4.18}$$

Then we have

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P}[A_t] \leq -b, \tag{4.19}$$

where

$$b := I_1 \left(-\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} \right) + I_2 \left(-\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2} \right) > 0.$$

Proof. The following holds:

$$\begin{aligned}
\ln L_t^1 &= -\int_0^t \psi_1(s) dW_s^1 - \frac{1}{2} \int_0^t \psi_1^2(s) ds \\
&= -\int_0^t \eta_1 \sqrt{r_s} \left(\frac{dr_s - (\kappa_1 \mu_1 - \kappa_1 r_s) ds}{\sigma_1 \sqrt{r_s}} \right) - \frac{1}{2} \int_0^t \eta_1^2 r_s ds \\
&= \frac{\eta_1}{\sigma_1} r_0 - \frac{\eta_1}{\sigma_1} r_t - \left(\frac{\eta_1 \kappa_1}{\sigma_1} + \frac{\eta_1^2}{2} \right) \int_0^t r_s ds + \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} t \\
&= \frac{\eta_1}{\sigma_1} r_0 + \left(\alpha_1 r_t + \alpha_2 \int_o^t r_s ds \right) + \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} t,
\end{aligned}$$

where

$$\alpha_1 := \frac{-\eta_1}{\sigma_1}, \quad \alpha_2 := -\left(\frac{\eta_1 \kappa_1}{\sigma_1} + \frac{\eta_1^2}{2} \right), \quad \eta_1 \in \left(-\frac{2\kappa_1}{\sigma_1}, 0 \right),$$

so that $\alpha_2 > 0$. Which implies that

$$\begin{aligned}
&\mathbb{P}\left[L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}} \right] \\
&= \mathbb{P}\left[\ln L_t^1 \geq -\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1} \right] \\
&= \mathbb{P}\left[\left(\alpha_1 r_t + \alpha_2 \int_o^t r_s ds \right) + \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} t \geq -\lambda_1 t \right] \\
&= \mathbb{P}\left[\frac{1}{t} \left(\alpha_1 r_t + \alpha_2 \int_o^t r_s ds \right) \geq -\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} \right].
\end{aligned} \tag{4.20}$$

Using a similar argument

$$\ln L_t^2 = \frac{\eta_2}{\sigma_2} \delta_0 + \left(\gamma_1 \delta_t + \gamma_2 \int_o^t \delta_s ds \right) + \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2} t,$$

and so

$$\mathbb{P}\left[L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}}\right] = \mathbb{P}\left[\frac{1}{t}\left(\gamma_1 \delta_t + \gamma_2 \int_o^t \delta_s ds\right) \geq -\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2}\right] \tag{4.21}$$

with

$$\gamma_1 := \frac{-\eta_2}{\sigma_2}, \quad \gamma_2 := -\left(\frac{\eta_2 \kappa_2}{\sigma_2} + \frac{\eta_2^2}{2}\right), \quad \eta_2 \in \left(-\frac{2\kappa_2}{\sigma_2}, 0\right),$$

so that $\gamma_2 > 0$.

Thus,

$$\mathbb{P}[A_t] = \mathbb{P}\left[L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}}\right]\mathbb{P}\left[L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}}\right]$$

by independence. It follows that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}[A_t] \\
&= \limsup_{t\to\infty} \frac{1}{t} \ln \left(\mathbb{P}\left[L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}}\right]\mathbb{P}\left[L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}}\right]\right) \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left[L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}}\right] \\
&\quad + \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left[L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}}\right] \\
&= \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left[\frac{1}{t}\left(\alpha_1 r_t + \alpha_2 \int_o^t r_s ds\right) \geq -\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1}\right] \\
&\quad + \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left[\frac{1}{t}\left(\gamma_1 \delta_t + \gamma_2 \int_o^t \delta_s ds\right) \geq -\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2}\right].
\end{aligned} \tag{4.22}$$

Next, since we chose $\lambda_2 \leq \mu_2 \eta_2^2/2$ and $\lambda_1 \leq \mu_1 \eta_1^2/2$, which imply

$$-\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2} \geq \mu_2 \gamma_2, \quad -\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} \geq \mu_1 \alpha_2.$$

And since from the proof of Lemma 4.1 above, the rate function $I_2 := \Lambda^*$ for the process Y_t is strictly increasing on $[\mu_2 \gamma_2, \infty)$ (and similarly, the rate function I_1 for the process X_t would be strictly increasing in $[\mu_1 \alpha_2, \infty)$), then applying the upper bound LDP inequalities in (4.10) with appropriate choice of closed sets G, we get

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}[A_t] \leq -I_1\left(-\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1}\right) - I_2\left(-\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2}\right) = -b \tag{4.23}$$

with

$$b := I_1\left(-\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1}\right) + I_2\left(-\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2}\right).$$

This ends the proof. □

Proposition 4.2. *Consider again the set*

$$A_t := \left\{ L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}} \right\} \cap \left\{ L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}} \right\}$$

for any time $t > 0$ as in (4.18)*. Then for $\lambda_1, \lambda_2 > 0$ verifying* (4.17)*, we have*

$$\mathbb{P}[A_t] \leq e^{-bt}, \quad \mathbb{Q}[A_t] \geq 1 - 2e^{-\lambda t} \tag{4.24}$$

for all $\mathbb{Q} \in \mathcal{M}_t^e$, where $\lambda := \min(\lambda_1, \lambda_2)$.

Proof. The first inequality in (4.24) is straightforward from Proposition 4.1 where

$$b = I_1 \left(-\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} \right) + I_2 \left(-\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2} \right) > 0.$$

Next, Radon-Nikodym theorem (see [19, Section 1.6]) yields, for all time $t > 0$,

$$\begin{aligned} \mathbb{Q}[A_t^c] &= \int_{A_t^c} L_t d\mathbb{P} = \int_{\{L_t^1 \geq e^{-\lambda_1 t}\}^c \cup \{L_t^2 \geq e^{-\lambda_2 t}\}^c} L_t d\mathbb{P} \\ &\leq \int_{\{L_t^1 \geq e^{-\lambda_1 t}\}^c} L_t d\mathbb{P} + \int_{\{L_t^2 \geq e^{-\lambda_2 t}\}^c} L_t d\mathbb{P} \\ &\leq e^{-\lambda_1 t} + e^{-\lambda_2 t} \leq 2e^{-\lambda t}, \end{aligned}$$

which implies that $\mathbb{Q}[A_t] \geq 1 - 2e^{-\lambda t}$, for all $\mathbb{Q} \in \mathcal{M}_t^e$, for the second inequality in (4.24), as required. □

Proof of Theorem 4.1. From Proposition 4.2 above, for a large time $t > 0$, we have

$$\mathbb{P}[A_t] \leq e^{-bt} =: \varepsilon_1, \quad \mathbb{Q}[A_t] \geq 1 - 2e^{-\lambda t} =: 1 - \varepsilon_2, \quad \forall \mathbb{Q} \in \mathcal{M}_t^e.$$

Since $\varepsilon_1, \varepsilon_2 > 0$, then for large enough time $t > 0$, [8, Proposition 2.3] implies that, there is a value process $V_t^{\varphi'} \in \mathcal{K}_t$, i.e. there is an admissible self-financing strategy $\varphi' \in \mathcal{H}$ such that

$$V_t^{\varphi'} \geq -2e^{-\lambda t} \quad \mathbb{P} - \text{a.s.}, \quad \text{and} \quad \mathbb{P}\left[V_t^{\varphi'} \geq 1 - 2e^{-\lambda t} \right] \geq 1 - e^{-bt}. \tag{4.25}$$

Next, for any positive constant $c < \lambda$ and for a large enough time $t > 0$, consider the trading strategy $\varphi_s := \varphi_s' e^{(\lambda - c)t}, s > 0$. Then, since $\varphi' \in \mathcal{H}$ so is φ, and we have (at large time t)

$$V_t^{\varphi} = \frac{1}{2} e^{(\lambda - c)t} V_t^{\varphi'}.$$

It follows we have both $V_t^{\varphi} \geq -e^{-ct}$ $\mathbb{P} - a.s.$, and

$$\begin{aligned} \mathbb{P}\left[V_t^{\varphi} \geq e^{ct} \right] &\geq \mathbb{P}\left[V_t^{\varphi} \geq \frac{1}{2} e^{(\lambda - c)t} - e^{-ct} \right] \\ &= \mathbb{P}\left[V_t^{\varphi'} \geq 1 - 2e^{-\lambda t} \right] \geq 1 - e^{-bt} \end{aligned} \tag{4.26}$$

for large enough time $t > 0$. Showing existence of asymptotic exponential arbitrage with geometrically decaying failure probability in the Liu-Tang 3-factor commodity futures model. □

Remark 4.1 (The Main Theorem in the General Case)**.** For computation simplicity, we proved Propositions 4.1 and 4.2 by assuming the constants $v_0 = v_{Sr} = v_{S\delta} = 0$. But in general these propositions and hence the main theorem (Theorem 4.1) hold true for the case where $v_0, v_{Sr}, v_{S\delta} > 0$ by modifying that: for $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, the set A_t to be defined by

$$A_t := \left\{ L_t^1 \geq e^{-\lambda_1 t - \frac{\eta_1 r_0}{\sigma_1}} \right\} \cap \left\{ L_t^2 \geq e^{-\lambda_2 t - \frac{\eta_2 \delta_0}{\sigma_2}} \right\} \cap \left\{ L_t^3 \geq e^{-(\lambda_3+\lambda_4)t - \beta} \right\} \tag{4.27}$$

for some constant β, where $\{L_t^1 \geq e^{-\lambda_1 t}\}, \{L_t^2 \geq e^{-\lambda_2 t}\}$ and $\{L_t^3 \geq e^{-(\lambda_3+\lambda_4)t}\}$ are independent events in $\mathcal{F}$, with

$$L_t^3 := \exp\left(-\int_0^t \psi_3(s) dW_s^3 - \frac{1}{2} \int_0^t \psi_3^2(s) ds \right),$$

and by similar arguments used in Proposition 4.1, one gets the constant b in this proposition to be replaced by

$$\begin{aligned} b := I_1 \left(-\lambda_1 - \frac{\eta_1 \kappa_1 \mu_1}{\sigma_1} \right) + I_1 \left(\frac{1}{2} (\eta_3 v_0 + \eta_3^2 v_0) + \frac{\eta_3 \sigma_{Sr} \kappa_1 \mu_1}{\sigma_1} - \lambda_3 \right) \\ + I_2 \left(-\lambda_2 - \frac{\eta_2 \kappa_2 \mu_2}{\sigma_2} \right) + I_2 \left(\frac{\eta_3 \sigma_{S\delta} \kappa_2 \mu_2}{\sigma_2} - \lambda_4 \right) > 0. \end{aligned}$$

5 Conclusion and perspectives

The work in this paper present a first novelty that is a generalization of the analysis of asymptotic exponential arbitrage discussed in [20] for the Schwartz one-factor commodity futures model where the convenience yield was constant and not a stochastic process, and the commodity futures model was complete (i.e. with a unique equivalent martingale measure) unlike its counterpart in the present 3-factor setting.

However, we notice that like in [20] and unlike the inspiring works [16,17] done in standard security markets, the trading opportunity generating asymptotic exponential arbitrage within the Liu-Tang 3-factor commodity futures model is not explicitly constructed. Nevertheless the result offers a hint to potential investors in the sense that even if arbitrage opportunities are ruled out for finite delivery date $T > 0$ as guaranteed in Proposition 3.3, trading in log-term (i.e. when T is large enough) in a futures exchange whose prices follow the Liu-Tang 3-factor commodity model may generate exponential growth risk-less profit for some trading strategy still to be found.

The present works open possible future research including interesting an case where the prices of the underlying storable commodity exhibits jumps and/or seasonal trends.

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