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Square-Mean Pseudo Almost Periodic Solutions of Infinite Class in the α-Norm under the Light of Measure Theory

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Abstract. This work concerns the existence and uniqueness of square-mean pseudo almost periodic solutions of infinite class in the α-norm. The results are obtained using analytic semigroup, fractional α-power theory and by making use of Ba-nach fixed point theory. As a result, we obtain a generalization of the work of Zab-sonre *et al.* [Partial Differential Equations and Applications: Colloquium in Honor of Hamidou Touré, Springer, 2023] in the deterministic case, without unbounded delay. Our results extend and complement many other important results in the lit-erature. Finally, a concrete example is given to illustrate the application of the main results.

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1 Introduction

In this work, we study the existence and uniqueness of square-mean (μ,ν)-pseudo almost periodic solutions of infinite class in the α-norm for the following stochastic evolution differential equation:

$$dx(t) = [-Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \quad t \in \mathbb{R}, \tag{1.1}$$

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where $-A : D(A) \subset H$ is the infinitesimal generator of compact analytic semigroup $(T(t))_{t\geq 0}$ on $L^2(P,H)$. The phase space $\mathcal{B}_\alpha$ defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \ \varphi(\theta) \in D(A^\alpha) \ \text{ for } \ \theta \leq 0 \ \text{ and } \ A^\alpha\varphi \in \mathcal{B}\}, \quad \|\varphi\|_\alpha = \|A^\alpha\varphi\|,$$

is a subset of $\mathcal{B}$, where $A^\alpha\varphi$ is defined by $A^\alpha\varphi(\theta) = A^\alpha(\varphi(\theta))$ for $\theta \in]-\infty,0]$ and $\mathcal{B}$ is a Banach space of functions mapping $]-\infty,0]$ into $L^2(P,H)$ and satisfying some axioms that will be presented later. A^α is the fractional α-power of A that will be described later. For every $t \geq 0$, the history function $u_t \in \mathcal{B}_\alpha$ is defined by

$$u_t(\theta) = u(t+\theta), \quad \theta \in]-\infty,0].$$

L is a bounded linear operator from $\mathcal{B}_\alpha$ into $L^2(P,H)$.

Here $f : \mathbb{R} \to L^2(P,H)$ and $g : \mathbb{R} \to L^2(P,H)$ are two stochastic processes and $W(t)$ is a two-sided standard Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ with

$$\mathcal{F}_t = \sigma\{W(u) - W(v)|u,v \leq t\}.$$

We assume $(H,\|,\|)$ is a real separable Hilbert space and $L^2(P,H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 dP < \infty.$$

Recall that stochastic modeling is crucial to many fields such as physics, engineering, economics, and social sciences. To this end, stochastic differential systems have been the subject of much research in recent years. Researchers are increasingly interested in the above mentioned quantitative and qualitative aspects of stochastic differential systems, such as existence, uniqueness, and stability. To this end, some recent contributions have been made concerning square-mean pseudo almost periodic for abstract differential equations similar to Eq. (1.1), see, for example, [4, 5, 9] and the references therein.

The aim of this work is to extend the results obtained by Zabsonre *et al.* [18], whose authors studied the Eq. (1.1) in the deterministic case. Note that some recent contributions have been made. For example, in [10], the authors studied the equation without operator L. They introduced a new concept of square-mean pseudo-almost periodic and automorphic processes using measure theory. They used the μ-ergodic process to define the spaces of μ pseudo almost periodic and automorphic processes in the square-mean sense. Moreover, they established many interesting results on these spaces, such as completeness and composition theorems. Then they studied the existence, the uniqueness and the stability of the square-mean μ-pseudo almost periodic and automorphic solutions of the stochastic evolution equation.

Recently, in [17], the authors, studied the existence and the uniqueness of the square-mean (μ,ν)-pseudo almost periodic solutions of infinite class for the stochastic evolution equation. However, to the best of the authors knowledge, the existence of square-mean (μ,ν)- pseudo almost periodic solutions of infinite class in the α-norm of the Eq. (1.1) remains untreated in the literature, which is the main motivation of this paper.

This paper is organized as follows. In Section 2, we recall some preliminary results about analytic semigroup and fractional power associated to its generator, in Section 3, we give the spectral decomposition of the phase space, in Section 4, we study the square-mean (μ,ν)-ergodic process of infinite class, in Section 5 we study square-mean (μ,ν)-pseudo almost periodic process, in Section 6, we discuss the existence and uniqueness of the square-mean (μ,ν)-pseudo almost periodic solution of infinite class. The last section is devoted to an application.

2 Analytic semigroup

Let $(L^2(P,H),\|\cdot\|)$ be a Banach space, let α be a constant such that $0<\alpha<1$ and let $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on $L^2(P,H)$. We assume without loss of generality that $0\in\rho(A)$. Note that if the assumption $0\in\rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A-\sigma I)$ with σ large enough such that $0\in\rho(A-\sigma I)$. This allows us to define the fractional power A^α for $0<\alpha<1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in $L^2(P,H)$. The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x|=\|x\|+\|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|\cdot|$ is equivalent to the norm $|x|_\alpha=\|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|\cdot|_\alpha$, is a Banach space, which we denote by $L^2(P,H_\alpha)$.

$(\mathbf{H}_0)$: The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on Banach space X. Moreover, we assume that $0\in\rho(A)$.

Proposition 2.1 ([14])**.** *Let $0<\alpha<1$ and assume that $(\mathbf{H}_0)$ hold. The following properties hold:*

i) $T(t):L^2(P,H)\to D(A^\alpha)$ *for every* $t>0$.

ii) $T(t)A^\alpha x=A^\alpha T(t)x$ *for every* $x\in D(A^\alpha)$ *and* $t\geq 0$.

iii) For every $t>0$, $A^\alpha T(t)$ *is bounded on* X *and there exist* $M_\alpha>0$ *and* $\omega>0$ *such that*

$$\|A^\alpha T(t)\|\leq M_\alpha e^{-\omega t}t^{-\alpha},\quad t>0.$$

iv) If $0<\alpha\leq\beta<1$, *then* $D(A^\beta)\hookrightarrow D(A^\alpha)$.

v) There exists $N_\alpha>0$ *such that*

$$\|(T(t)-I)A^{-\alpha}\|\leq N_\alpha t^\alpha,\quad t>0.$$

Recall that $A^{-\alpha}$ is given by the following formula:

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$. Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

3 Spectral decomposition

In this work, we assume that the state space $(\mathcal{B}, |.|_{\mathcal{B}})$ is a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following fundamental axioms.

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty, a] \to L^2(P, H)$, $u_\sigma \in \mathcal{B}$, and $u(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, a]$ the following conditions hold:

(i) $u_t \in \mathcal{B}$,

(ii) $|u(t)| \leq H|u_t|$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,

(iii) $|u_t| \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |u(s)| + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

$(\mathbf{A}_2)$ For the function $u(.)$ in $(\mathbf{A}_1), t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

$(\mathbf{B})$ The space $\mathcal{B}$ is a Banach space.

We make the following assumption:

$(\mathbf{H}_1)$ $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, the function $A^{-\alpha}\varphi$ is defined by

$$(A^{-\alpha}\varphi)(\theta) = A^{-\alpha}\varphi(\theta).$$

Consequently, we get the following result.

Lemma 3.1 ([11]). *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. If* $\mathcal{B}$ *satisfies axioms* $(\mathbf{A}_1), (\mathbf{A}_2)$ *and* $(\mathbf{B})$. *Then* $\mathcal{B}_\alpha$ *satisfies axioms* $(\mathbf{A}_1), (\mathbf{A}_2)$ *and* $(\mathbf{B})$.

$(\mathbf{A}_1)$ *If* $u :]-\infty, a] \to L^2(P, H_\alpha)$ *is continuous on* $[\sigma, a]$ *with* $x_\sigma \in \mathcal{B}_\alpha$, *for some* $\sigma < a$, *then for all* $t \in [\sigma, a]$:

(i) $u_t \in \mathcal{B}_\alpha$,

(ii) $\|u(t)\|_\alpha \leq H\|u_t\|_\alpha$, *which is equivalent to* $\|\varphi(0)\|_\alpha \leq H\|\varphi\|_\alpha$ *for every* $\varphi \in \mathcal{B}_\alpha$,

(iii) $\|u_t\|_\alpha \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} \|u(s)\|_\alpha + M(t-\sigma)\|u_\sigma\|_\alpha$.

$(\mathbf{A}_2)$ *For the function* $u(.)$ *in* $(\mathbf{A}_1)$, $t \to u_t$ *is* $\mathcal{B}_\alpha$*-valued continuous function for* $t \in [\sigma, \sigma + a]$.

$(\mathbf{B})$ *The space* $\mathcal{B}_\alpha$ *is a Banach space.*

We suppose that the phase space $\mathcal{B}$ satisfies the following axiom:

$(\mathbf{C}_1)$ If $(\varphi_n)_{n\geq 0}$ is the Cauchy sequence in $\mathcal{B}$ such that $\varphi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then, $(\varphi_n(\theta))_{n\geq 0}$ converges to 0 in X.

Let $C(]-\infty, 0], L^2(P,H))$ be the space of continuous functions from $]-\infty, 0]$ into $L^2(P,H)$. We suppose the following assumptions hold:

$(\mathbf{C}_2)$ $\mathcal{B} \subset C(]-\infty, 0], L^2(P,H))$,

$(\mathbf{C}_3)$ there exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with Re $\lambda > \lambda_0$ and $x \in L^2(P,H)$ we have $e^\lambda x \in \mathcal{B}$, and

$$K_0 = \sup_{\substack{\operatorname{Re}\lambda > \lambda_0,\ x \in L^2(P,H),\\ x \neq 0}} \frac{|e^\lambda x|_{\mathcal{B}}}{|x|} < \infty,$$

where $(e^\lambda x)(\theta) = e^{\lambda\theta} x$ for $\theta \in]-\infty, 0]$ and $x \in L^2(P,H)$.

To Eq. (1.1), we associate the following initial value problem:

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t), \quad t \geq 0, \\ u_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{3.1}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function. For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on $\mathcal{B}_\alpha$ by

$$\mathcal{U}(t) = v_t(., \varphi),$$

where $v(., \varphi)$ is the solution of the following homogeneous equation:

$$\begin{cases} \dfrac{d}{dt}v(t) = -Av(t) + L(v_t), \quad t \geq 0, \\ v_0 = \varphi \in \mathcal{B}_\alpha. \end{cases}$$

Proposition 3.1 ([3]). *Assume that* $\mathcal{B}$ *satisfies* $(\mathbf{A}_1), (\mathbf{A}_2), (\mathbf{B}), (\mathbf{C}_1)$ *and* $(\mathbf{C}_2)$, *then the generator* $\mathcal{A}_\mathcal{U}$ *of* $(\mathcal{U}(t))_{t\geq 0}$ *is defined on* $\mathcal{B}_\alpha$ *by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) = \{\varphi \in \mathcal{B}_\alpha, \varphi' \in \mathcal{B}_\alpha, \varphi(0) \in D(A), \varphi(0)' \in \overline{D(A)} \text{ and } \varphi(0)' = -A\varphi(0) + L(\varphi)\}, \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \in D(\mathcal{A}_\mathcal{U}). \end{cases}$$

Then $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_t \geq 0$ on $\mathcal{B}_\alpha$.

Let $\langle X_0 \rangle$ be the space defined by $\langle X_0 \rangle = \{X_0 c : c \in L^2(P,H)\}$, where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0, & \text{if} \quad \theta \in]-\infty, 0[, \\ c, & \text{if} \quad \theta = 0. \end{cases}$$

The space $\mathcal{B}_\alpha \oplus \langle X_0 \rangle$ endowed with the norm

$$\|\phi + X_0 c\|_\alpha = \|\phi\|_\alpha + \|c\|, \quad (\phi, c) \in \mathcal{B}_\alpha \times L^2(P, H)$$

is Banach space. Consider the extension $\widetilde{\mathcal{A}_\mathcal{U}}$ of $\mathcal{A}_\mathcal{U}$ defined on $\mathcal{B}_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \{\varphi \in \mathcal{B}_\alpha, \varphi(0) \in D(A), \varphi'(0) \in \overline{D(A)} \text{ and } \varphi(0)' \in \overline{D(A)}\}, \\ \widetilde{\mathcal{A}}_\mathcal{U}\varphi = \varphi' + X_0\big(-A\varphi(0) + L(\varphi) - \varphi(0)'\big). \end{cases}$$

Lemma 3.2 ([2]). *Assume that $\mathcal{B}$ satisfies $(\mathbf{A}_1), (\mathbf{A}_2), (\mathbf{B}), (\mathbf{C}_1)$, and $(\mathbf{C}_2)$, and $(\mathbf{H}_0)$ hold. Then $\mathcal{A}_\mathcal{U}$ satisfies the Hile-Yosida condition on $\mathcal{B}_\alpha \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0, \widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_\mathcal{U}})$ and*

$$\big\|(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-n}\big\|_\alpha \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n}, \quad n \in \mathbb{N}, \quad \lambda > \widetilde{\omega}.$$

Let C_{00} be the space of X-valued continuous function on $]-\infty, 0]$ with compact support. We assume that

$(\mathbf{D})$ If φ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to φ on $]-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $|\varphi_n - \varphi| \to 0$.

Proposition 3.2 ([3]). *The family $(\mathcal{U}(t))_{t\geq 0}$ is a strongly semigroup on $\mathcal{B}_\alpha$, that is*

(i) $\mathcal{U}(0) = I$,

(ii) $\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$ *for* $t, s \geq 0$,

(iii) for all $\varphi \in \mathcal{B}_\alpha$, $\mathcal{U}(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* $\mathcal{B}_\alpha$,

(iv) $\mathcal{U}(t)$ *satisfies the translation property, that is for* $t \geq 0$ *and* $\theta \leq 0$, *one has*

$$\big(\mathcal{U}(t)(\varphi)\big)(\theta) = \begin{cases} \big(\mathcal{U}(t+\theta)(\varphi)\big)(0), & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0. \end{cases}$$

For $\varphi \in \mathcal{B}$ and $\theta \leq 0$, we define the linear operator W by

$$[W(t)\varphi](\theta) = \begin{cases} \varphi(0), & \text{if } \quad t+\theta \geq 0, \\ \varphi(t+\theta), & \text{if } \quad t+\theta < 0. \end{cases}$$

$(W(t))_{t\geq 0}$ is exactly the solution semigroup associated to the following equation:

$$\begin{cases} \dfrac{d}{dt}u(t) = 0, \\ u_0 = 0. \end{cases}$$

Let $W_0(t) = W(t)_{/\widetilde{\mathcal{B}}}$, where $\widetilde{\mathcal{B}} := \{\varphi \in \mathcal{B} : \varphi(0) = 0\}$.

Definition 3.1 ([3]). *$\mathcal{B}$ is called a uniform fading memory space if it satisfies axioms $\mathcal{B}$ satisfies $(\mathbf{A}_1), (\mathbf{A}_2), (\mathbf{B}), (\mathbf{C}_1)$ and $(\mathbf{C}_2), (\mathbf{D})$ and $\|W_0(t)\| \to 0$ as $t \to +\infty$.*

Lemma 3.3 ([13]). *If $\mathcal{B}$ is uniform fading memory space, then we can choose the function $K()$ and the function $M(\cdot)$ such that $M(t) \to 0$ as $t \to +\infty$.*

Proposition 3.3 ([13]). *If the phase space $\mathcal{B}$ is a fading memory space, then the space $BC(]-\infty,0], L^2(P,H)]$ of bounded continuous $L^2(P,H)$-valued functions on $]-\infty,0]$ endowed with the uniform norm topology is continuously embedded in $\mathcal{B}$. In particular $\mathcal{B}$ satisfies $(\mathbf{C}_3)$ for $\lambda_0 > 0$.*

Definition 3.2 ([2]). *We say a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \varnothing.$$

We make the following assumption.

$(\mathbf{H}_2)$ $T(t)$ is compact on $L^2(P,H)$ for each $t > 0$.

We have the following result on the spectral decomposition of the phase space $\mathcal{B}_\alpha$.

Theorem 3.1 ([11]). *Assume that $(\mathbf{H}_0), (\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. Suppose that $\mathcal{B}$ is a uniform fading memory space. Then $\mathcal{B}_\alpha$ is decomposed as a direct sum*

$$\mathcal{B}_\alpha = S \oplus V$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t\geq 0}$ on U is a group and there exist positive constants $\overline{M}$ and ω such that

$$\begin{aligned} \|\mathcal{U}(t)\varphi\|_\alpha &\leq \overline{M}e^{-\omega t}\|\varphi\|_\alpha, \quad t \geq 0, \quad \varphi \in S, \\ \|\mathcal{U}(t)\varphi\|_\alpha &\leq \overline{M}e^{\omega t}\|\varphi\|_\alpha, \quad t \leq 0, \quad \varphi \in U, \end{aligned}$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

4 Square mean (μ,ν)-ergodic process of infinite class

In the following $\mathcal{N}$ denotes the Lebesgue σ-field of $\mathbb{R}$, $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$. Let $L^2(P,H)$ is a Hilbert space endowed with the following norm:

$$\|x\|_{L^2} = \left(\int_\Omega \|x\|^2 dP \right)^{\frac{1}{2}}.$$

Definition 4.1 ([10]). *Let $x : \mathbb{R} \to L^2(P,H)$ be a stochastic process.*

1. *x said to be stochastically bounded in square-mean sense, if there exists $M > 0$ such that*

$$\mathbb{E}\|x(t)\|^2 \leq M, \quad \forall\, t \in \mathbb{R}.$$

2. *x said to be stochastically continuous in square-mean sense if*

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|^2 \leq M, \quad \forall\, t, s \in \mathbb{R}.$$

Denote by $SBC(\mathbb{R}, L^2(P, H))$ the space of all the stochastically bounded continuous processes.

Remark 4.1 ([10]). $(SBC(\mathbb{R}, L^2(\Omega, H)), \|\cdot\|_\infty)$ is a Banach space, where

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \left(\mathbb{E}(\|x(t)\|^2)\right)^{\frac{1}{2}}.$$

Definition 4.2. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be α-(μ, ν)-ergodic in square-mean sense, if $f \in BC(\mathbb{R}, L^2(P, H_\alpha))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|_\alpha^2 d\mu(t) = 0.$$

We denote by $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu)$, the space of all such process.

Definition 4.3. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be α-(μ, ν)-ergodic of infinite class in square-mean sense, if $f \in BC(\mathbb{R}, L^2(P, H_\alpha))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty, t]} \mathbb{E}\|f(t)\|_\alpha^2 d\mu(t) = 0.$$

We denote by $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$, the space of all such process. For $\mu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote μ_a the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$\mu_a(A) = \mu([a + b : b \in A]), \quad A \in \mathcal{N}. \tag{4.1}$$

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypotheses for our results.

($\mathbf{H}_2$) Let $\mu, \nu \in \mathcal{M}$ be such that

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \delta < \infty.$$

($\mathbf{H}_3$) For all a, b and $c \in \mathbb{R}$ such that $0 \leq a < b < c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \;\Rightarrow\; \mu(a + \delta, b + \delta) \geq \alpha_0 \mu(\delta, c + \delta).$$

($\mathbf{H}_4$) For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta \mu(A),$$

when $A \in \mathcal{N}$ and satisfies $A \cap I = \varnothing$.

Proposition 4.1. *Assume that* $(\mathbf{H}_2)$ *holds. Then space* $\mathcal{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ *endowed with uniform topology norm is a Banach space.*

Proof. Note that $\mathcal{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ is a vector subspace of $SBC(\mathbb{R}; L^2(P, H_\alpha))$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}; L^2(P, H_\alpha), \mu, \nu, \infty)$ is closed in $SBC(\mathbb{R}, L^2(P, H_\alpha))$. Let $(f_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ such that $\lim_{n\to+\infty} f_n = f$ uniformly in $\mathbb{R}$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau, \tau]) > 0$ for τ sufficiently large. Let

$$\|f\|^2_{\infty,\alpha} = \sup_{t\in\mathbb{R}} \mathbb{E}\|f(t)\|^2_\alpha,$$

and $n_0 \in \mathbb{N}$ such that all $n \geq n_0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in]-\infty,t]} E\|f(\theta)\|^2_\alpha\right)d\mu(t)\\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in]-\infty,t]} \mathbb{E}\|f_n(\theta)-f(\theta)\|^2_\alpha\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in]-\infty,t]} \mathbb{E}|f_n(\theta)\|^2_\alpha\right)d\mu(t)\\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in\mathbb{R}} \mathbb{E}\|f_n(\theta)-f(\theta)\|^2_\alpha\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in]-\infty,t]} \mathbb{E}|f_n(\theta)\|^2_\alpha\right)d\mu(t)\\
&\leq \|f_n-f\|^2_{\infty,\alpha}\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in]-\infty,t]} \mathbb{E}\|f_n(\theta)\|^2_\alpha\right)d\mu(t),
\end{aligned}$$

which implies that

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in]-\infty,t]} \mathbb{E}|f(\theta)\|^2_\alpha\right)d\mu(t) \leq \delta\varepsilon, \quad \forall\, \varepsilon > 0.$$

The proof is complete. □

The following theorem is a characterization of square-mean α-(μ,ν)-ergodic processes.

Theorem 4.1. *Assume that* $(\mathbf{H}_2)$ *holds and let* $\mu, \nu \in \mathcal{M}$ *and* I *be a bounded interval (eventually* $I = \varnothing$*). Assume that* $f \in BC(\mathbb{R}, L^2(P, H_\alpha))$*. The following assertions are equivalent:*

(i) $f \in \mathcal{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$.

(ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\left(\sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|^2_\alpha\right)d\mu(t) = 0.$

(iii) $\forall\, \varepsilon > 0, \quad \lim_{\tau \to +\infty} \frac{\mu\big(\{t \in [-\tau,\tau] \setminus I : \mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\nu([-\tau,\tau] \setminus I)} = 0.$

Proof. The proof uses the same arguments of the proof of [8, Theorem 2.22].

$(i) \Leftrightarrow (ii)$. Denote by

$$A = \mu(I), \quad B = \int_I \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) d\mu(t).$$

Since the interval I is bounded and the process f is stochastically bounded continuous, A and B are finite. For $\tau > 0$ such that $I \subset [-\tau,\tau]$ and $\nu([-\tau,\tau] \setminus I) > 0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta\|_\alpha^2 \right) d\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau]) - A} \left[\int_{[-\tau,\tau]} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) d\mu(t) - B \right] \\
&= \frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau]) - A} \left[\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}|f(\theta)\|_\alpha^2 \right) d\mu(t) - \frac{B}{\nu([-\tau,\tau])} \right].
\end{aligned}$$

From above equalities and the fact $\nu(\mathbb{R}) = +\infty$, we deduce (ii) is equivalent to

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) d\mu(t) = 0,$$

that (i).

$(iii) \Rightarrow (ii)$. Denote by A_τ^ε and B_τ^ε the following sets:

$$A_\tau^\varepsilon = \left\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon \right\},$$

$$B_\tau^\varepsilon = \left\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \le \varepsilon \right\}.$$

Assume that (ii) holds, that is

$$\lim_{\tau \to +\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]) \setminus I)} = 0. \tag{4.2}$$

From the equality

$$\begin{aligned}
&\int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) d\mu(t) \\
&= \int_{A_\tau^\varepsilon} \left(\sup_{\theta \in]-\infty,t]} \|f(\theta)\|^p \right) d\mu(t) + \int_{B_\tau^\varepsilon} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) d\mu(t),
\end{aligned}$$

we deduce that for τ sufficient large,

$$\begin{aligned}&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu(t)\\ &\leq \|f\|_{\infty,\alpha}^2\times\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}.\end{aligned}$$

Since $\mu(\mathbb{R})=\nu(\mathbb{R})=\infty$ and by using $(\mathbf{H}_2)$ then for all $\varepsilon>0$, we have

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu(t)\leq\delta\varepsilon.$$

Consequently, (ii) holds.

$(ii)\Rightarrow(iii)$.

$$\begin{aligned}&\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu(t)\geq\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu(t),\\ &\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu(t)\geq\varepsilon\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},\\ &\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu(t)\geq\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}\end{aligned}$$

for τ sufficiently large, we obtain Eq. (4.2), that is (iii). □

Definition 4.4. *Let $f\in SBC(\mathbb{R},L^2(P,H_\alpha))$ and $\tau\in\mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t)=f(t+\tau)$ for $t\in\mathbb{R}$. A subset $\mathscr{F}$ of $SBC(\mathbb{R},L^2(P,H_\alpha))$ is said to be translation invariant if for all $f\in\mathscr{F}$, we have $f_\tau\in\mathscr{F}$ for all $\tau\in\mathbb{R}$.*

Definition 4.5. *Let $\mu_1,\mu_2\in\mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoting this as $\mu_1\sim\mu_2$ if there exist constants α and $\beta>0$ and a bounded interval I (eventually $I=\varnothing$) such that $\alpha\mu_1(A)\leq\mu_2(A)\leq\beta\mu_1(A)$, when $A\in\mathcal{N}$ satisfies $A\cap I=\varnothing$.*

Remark 4.2. The relation $\sim$ is an equivalence relation on $\mathcal{M}$.

Theorem 4.2. *Let $\mu_1,\nu_1,\mu_2,\nu_2\in\mathcal{M}$. If $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$, then*

$$\mathscr{E}\big(\mathbb{R},L^2(P,H_\alpha),\mu_1,\nu_1,\infty\big)=\mathscr{E}\big(\mathbb{R},L^2(P,H_\alpha),\mu_2,\nu_2,\infty\big).$$

Proof. Since $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$, there exists some constants $\alpha_1,\alpha_2,\beta_1,\beta_2>0$ and a bounded interval I (eventually $I=\varnothing$) such that $\alpha_1\mu_1(A)\leq\mu_2(A)\leq\beta_1\mu_2(A)$ and $\alpha_2\nu_1(A)\leq\nu_2(A)\leq\beta_1\nu_1(A)$ for each $A\in\mathcal{N}$ satisfies $A\cap I=\varnothing$, i.e.

$$\frac{1}{\beta_2\nu_1(A)}\leq\frac{1}{\nu_2(A)}\leq\frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field for τ sufficiently large,

$$\begin{aligned}
&\frac{\alpha_1\mu_1\big(\{t \in [-\tau,\tau]\setminus I : \sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\beta_2\mu_2([-\tau,\tau]\setminus I)}\\
&\leq \frac{\mu_2\big(\{t \in [-\tau,\tau]\setminus I : \sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\nu_2([-\tau,\tau]\setminus I)}\\
&\leq \frac{\beta_1\mu_1\big(\{t \in [-\tau,\tau]\setminus I : \sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\alpha_2\nu([-\tau,\tau]\setminus I)}.
\end{aligned}$$

By using Theorem 4.1, we deduce that

$$\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu_1,\nu_1,\infty) = \mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu_2,\nu_2,\infty).$$

Let $\mu,\nu \in \mathcal{M}$, we denote

$$cl(\mu,\nu) = \{\overline{\omega}_1,\overline{\omega}_2 \in \mathcal{M} : \mu_1 \sim \mu_2, \nu_1 \sim \nu_2\}.$$

The proof is complete. □

Lemma 4.1 ([7]). *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_4)$. Then for all $\tau \in \mathbb{R}$ the measures μ and μ_τ are equivalent.*

Lemma 4.2 ([7]). $(\mathbf{H}_4)$ *implies*

$$\forall\, \sigma > 0, \quad \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma,\tau+\sigma])}{\nu([-\tau,\tau])} < \infty.$$

Theorem 4.3. *Assume that $(\mathbf{H}_4)$ holds. Then $\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,\infty)$ is invariant by translation.*

Proof. The proof is inspired by [6, Theorem 3.5]. Let $f \in \mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$, there exists $a_0 > 0$ such that $\nu([-\tau-|a|,\tau+|a|]) > 0$ for $|a| > a_0$. Denote

$$M_a(\tau) = \frac{1}{\nu_a([\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\Big)d\mu_a(t), \quad \forall\,\tau > 0, \quad a \in \mathbb{R},$$

where ν_a is the positive measure define by Eq. (4.1). By using Lemma 4.1, it follows that ν and ν_a are equivalent, μ and μ_a are equivalent and by Theorem 4.2, we have

$$\mathscr{E}\big(\mathbb{R}, L^2(P,H_\alpha),\mu_a,\nu_a,\infty\big) = \mathscr{E}\big(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,\infty\big),$$

therefore, $f \in \mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu_a,\nu_a,r)$ that is $\lim_{t\to+\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$. For all $A \in \mathcal{N}$, we denote χ_A the characteristic function of A. By using definition of the μ_a, we obtain

$$\int_{[-\tau,\tau]}\chi_A(t)d\mu_a(t) = \int_{[-\tau,\tau]}\chi_A(t)d\mu_a(t+a) = \int_{[-\tau+a,\tau+a]}\chi_A(t)d\mu_a(t).$$

Since $t \mapsto \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2$ is the pointwise limit of an increasing sequence of function (see [19, Theorem 1.17]), we deduce that

$$\int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu_a(t) = \int_{[-\tau+a,\tau+a]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)\|^2 d\mu(t).$$

We denote $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$. Then we have $|a| + a = 2a^+$, $|a| - a = 2a^-$ and $[-\tau + a - |a|, \tau + a + |a|] = [-\tau - 2a^-, \tau + 2a^+]$. Therefore, we obtain

$$M_a(\tau+|a|) = \frac{1}{\nu([-\tau-2a^-,\tau+2a^+])} \int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t). \quad (4.3)$$

From (4.3) and the following inequality:

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\ &\leq \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)|_\alpha^2 d\mu(t), \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\ &\leq \frac{\nu([-\tau-2a^-,\tau+2a^+])}{\nu([-\tau,\tau])} \times M_a(\tau+|a|). \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\ &\leq \frac{\nu([-\tau-2|a|,\tau+2|a|])}{\nu([-\tau,\tau])} \times M_a(\tau+|a|). \end{aligned} \quad (4.4)$$

From Eqs. (4.3) and (4.4), and using Lemma 4.2, we deduce that

$$\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) = 0,$$

which equivalent to

$$\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta-a)\|_\alpha^2 d\mu(t) = 0,$$

that is $f_a \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$. We have proved that $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ then $f_{-a} \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ for all $a \in \mathbb{R}$, which means $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ invariant by translation. □

5 Square mean (μ, ν)-pseudo almost periodic process

In this section, we define (μ, ν)-pseudo almost periodic and we study their basic properties.

Definition 5.1. *A continuous process $f : \mathbb{R} \to L^2(P, H_\alpha)$ is said to be α-square-mean almost periodic process, if for each $\varepsilon > 0$ there exists $l > 0$ such that for all $\beta \in \mathbb{R}$, there exists $\tau \in [\beta, \beta + l]$ with*

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|f(t+\tau) - f(t)\|_\alpha^2 < \varepsilon. \tag{5.1}$$

We denote the space of all such stochastic processes by $SAP(\mathbb{R}, L^2(P, H_\alpha))$.

Theorem 5.1 ([10])**.** *The space $SAP(\mathbb{R}, L^2(P, H))$ endowed with the norm $\|.\|_\infty$ is a Banach space.*

Definition 5.2. *Let $\mu, \nu \in \mathcal{M}$. A continuous process $f : \mathbb{R} \to L^2(P, H_\alpha)$ is said to be α-(μ, ν)-square-mean pseudo almost periodic process if it can decomposed as follows:*

$$f = g + \phi,$$

where $g \in SAP(\mathbb{R}, L^2(P, H_\alpha))$ and $\phi \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu)$. We denote the space of such stochastic processes by $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu)$.

Proposition 5.1. *Assume that $(\mathbf{H}_3)$ holds. Then the decomposition of α-(μ, ν)-pseudo almost periodic function in the form $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}, X_\alpha)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}, X_\alpha, \mu, \nu)$ is unique.*

Remark 5.1. Let $X = L^2(P, H_\alpha)$. Then the Proposition 5.1 always holds.

Definition 5.3. *Let $\mu, \nu \in \mathcal{M}$. A continuous process $f : \mathbb{R} \to L^2(P, H_\alpha)$ is said to be α-(μ, ν)-square-mean pseudo almost periodic process of infinite class if it can decomposed as follows:*

$$f = g + \phi,$$

where $g \in SAP(\mathbb{R}, L^2(P, H_\alpha))$ and $\phi \in \mathscr{E}(\mathbb{R}, L^2(P, H)_\alpha, \mu, \nu, \infty)$. We denote the space of such stochastic processes by $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$.

Proposition 5.2. *Assume that $(\mathbf{H}_3)$ holds. Then the decomposition of α-(μ, ν)-pseudo almost periodic process of infinite class in the form $\phi = \phi_1 + \phi_2$, where $\phi_1 \in SAP(\mathbb{R}, L^2(P, H_\alpha))$ and $\phi_2 \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$ is unique.*

Proposition 5.3. *Let μ_1, μ_2, ν_1 and $\nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then*

$$SPAP\big(\mathbb{R}, L^2(P, H_\alpha), \mu_1, \nu_1, \infty\big) = SPAP\big(\mathbb{R}, L^2(P, H_\alpha), \mu_2, \nu_2, \infty\big).$$

Proof. This proposition is just a consequence of Theorem 4.2. □

Theorem 5.2. *Assume that* $(\mathbf{H}_3)$ *holds . Let* $\mu,\nu\in\mathcal{M}$ *and* $\phi\in SPAP(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,\infty)$, *then* $t\to\phi_t$ *belongs to* $SPAP(\mathcal{B}_\alpha,\mu,\nu,)$.

Proof. Assume that $\phi=g+h$, where

$$g\in SAP\big(\mathbb{R},L^2(P,H_\alpha)\big),\quad h\in\mathscr{E}\big(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,\infty\big).$$

Then we can see that $\phi_t=g_t+h_t$ and g_t is square mean almost periodic process. Let us denote

$$M_a=\frac{1}{\nu_a([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2 d\mu_a(t),$$

where μ_a and ν_a are the positive measures defined by Eq. (4.1). By using Lemma 4.1, it follows that μ and μ_a are equivalent, ν and ν_a are equivalent by using Theorem 4.2,

$$\mathscr{E}\big(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,\infty\big)=\mathscr{E}\big(\mathbb{R},L^2(P,H_\alpha),\mu_a,\nu_a,\infty\big).$$

Therefore, $f\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu_a,\nu_a,\infty)$ that is $\lim_{\tau\to\infty}M_a(\tau)=0$ for all $a\in\mathbb{R}$. On the other hand for $\tau>0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\sup_{\theta\in]-\infty,0]}\mathbb{E}\|h(\theta+\xi)\|_\alpha^2\right)d\mu(t)\\
&\leq\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\leq\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t-r]}\mathbb{E}\|h(\theta)\|_\alpha^2+\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\leq\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t-r]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\leq\frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{\tau-r}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\leq\frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{\tau+}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t+r)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t]}\|h(\theta)\|_\alpha^2\right)d\mu(t)\\
&\leq\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\left(\frac{1}{\nu([-\tau-r,\tau+r])}\int_{-\tau-r}^{\tau}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)d\mu(t+r)\right)
\end{aligned}$$

$$+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\Big)d\mu(t).$$

Consequently,

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\sup_{\theta\in]-\infty,0]}\mathbb{E}\|h(\theta+\xi)\|_\alpha^2\Big)d\mu(t)$$
$$\leq \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\times M_r(\tau+r)+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\Big)d\mu(t),$$

which shows using Lemmas 4.1 and 4.2 that ϕ_t belongs to $SPAP(\mathcal{B}_\alpha,\mu,\nu,\infty)$. Thus, we obtain the desired result. □

Definition 5.4 ([10]). *Let $f:\mathbb{R}\times L^2(P,H_\alpha)\to L^2(P,H_\alpha),(t,x)\mapsto f(t,x)$ be continuous stochastic process. f is said α-square-mean in $t\in\mathbb{R}$ uniformly in $x\in L^2(P,H)$ if for all compact K of $L^2(P,H)$ and for any $\varepsilon>0$ there exists $l(\varepsilon,K)$ such that for all $\beta\in\mathbb{R}$, there exists $\tau\in[\beta,\beta+l(\varepsilon,K)]$ with*

$$x\in K,\quad \sup_{t\in\mathbb{R}}\mathbb{E}\|f(t+\tau,x)-f(t,x)\|_\alpha^2<\varepsilon.$$

We denote the following space of stochastic processes by $SAP(\mathbb{R}\times L^2(P,H_\alpha),L^2(P,H_\alpha))$.

Definition 5.5. *Let $\mu,\nu\in\mathcal{M}$. A continuous stochastic process $f:\mathbb{R}\times L^2(P,H_\alpha)\to L^2(P,H_\alpha)$ is said to be square-mean α-(μ,ν)-pseudo almost periodic if it can be written as*

$$f=g+\phi,$$

where $g\in SAP(\mathbb{R}\times L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}\times L^2(P,H_\alpha),\mu,\nu)$. We denote the following space of stochastic processes by $SPAP(\mathbb{R}\times L^2(P,H_\alpha),L^2(P,H_\alpha),\mu,\nu)$.

Definition 5.6. *Let $\mu,\nu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\times L^2(P,H_\alpha)\to L^2(P,H_\alpha)$ is said to be square-mean α-(μ,ν)-pseudo almost periodic of infinite class if it can be written as*

$$f=g+\phi,$$

where $g\in SAP(\mathbb{R}\times L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}\times L^2(P,H_\alpha),\mu,\nu,\infty)$. We denote the following space of stochastic processes by $SPAP(\mathbb{R}\times L^2(P,H_\alpha),L^2(P,H_\alpha),\mu,\nu,\infty)$.

Next, we study the composition of square-mean α-(μ,ν) pseudo almost periodic processes of infinite class.

Theorem 5.3 ([10]). *Let $f:\mathbb{R}\times L^2(P,H_\alpha)\to L^2(P,H)$, $(t,x)\mapsto f(t,x)$ be square mean almost periodic process in $t\in\mathbb{R}$ uniformly in $x\in L^2(P,H_\alpha)$. Suppose that f satisfies the Lipschitz condition in the following sense:*

$$\mathbb{E}\|f(t,x)-f(t,y)\|_\alpha^2\leq L\,\mathbb{E}\|x-y\|_\alpha^2$$

for all $x,y\in L^2(P,H_\alpha)$ and each $t\in\mathbb{R}$, where L is independent of t. Then $f(t,x(t))\in SAP(\mathbb{R},L^2(P,H_\alpha))$ for any $x\in SAP(\mathbb{R},L^2(P,H))$.

Theorem 5.4. *Let* $\mu, \nu \in \mathcal{M}, \phi = \phi_1 + \phi_2 \in SPAP(\mathbb{R} \times L^2(P, H_\alpha), \mu, \nu, \infty)$*, where* $\phi_1 \in SPAP(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H))$ *and* $\phi_2 \in \mathscr{E}(P(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H), \mu, \nu, \infty)$ *and* $h \in SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, \infty)$*. Assume:*

(i) $\phi_1(t, x)$ *is uniformly continuous on any bounded subset uniformly for* $t \in \mathbb{R}$*,*

(ii) there exists a nonnegative function $L_\phi \in L^p(\mathbb{R}), (1 \leq p \leq +\infty)$ *such that*

$$\mathbb{E}\|\phi(t, x_1) - \phi(t, x_2)\|^2 \leq L_\phi(t)\mathbb{E}\|x_1 - x_2\|_\alpha^2, \quad \forall\, t \in \mathbb{R}, \quad \forall\, x_1, x_2 \in L^2(P, H_\alpha). \quad (5.2)$$

If

$$\vartheta = \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in]-\infty, t]} \big(L_\phi(\theta)\big) d\mu(t) < \infty, \quad (5.3)$$

then the function $t \to \phi(t, h(t))$ *belongs to* $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$*.*

To prove this theorem, we need the following lemma.

Lemma 5.1. *Assume that* $(\mathbf{H}_3)$ *holds and let* $f \in SBC(\mathbb{R}, L^2(P, H))$*. Then*

$$f \in \mathscr{E}\big(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty\big),$$

if and only if

$$\lim_{t \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])} = 0,$$

where

$$M_{\tau,\varepsilon}(f) = \Big\{\tau \in [-\tau, \tau] : \sup_{\theta \in]-\infty, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \geq \varepsilon\Big\}.$$

Proof. Assume that $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. Then

$$\begin{aligned}
&\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in]-\infty, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\
&= \frac{1}{\nu([-\tau, \tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\
&\quad + \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\
&\geq \frac{1}{\nu([-\tau, \tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \\
&\geq \frac{\varepsilon \mu\big(M_{\tau,\varepsilon}(f)\big)}{\nu([-\tau, \tau])}.
\end{aligned}$$

Consequently,

$$\lim_{\tau \to +\infty} \frac{\mu\big(M_{\tau,\varepsilon}(f)\big)}{\nu([-\tau, \tau])} = 0.$$

Suppose that $f \in SBC(\mathbb{R}; L^2(P, H_{\alpha)}))$ such that for any $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu\big(M_{\tau,\varepsilon}(f)\big)}{\nu([-\tau,\tau])} = 0.$$

Assume that $\mathbb{E}\|f(t)\|_\alpha^2 \le N$ for all $t \in \mathbb{R}$, then using $(\mathbf{H}_3)$, it follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t)\\
&= \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t)\\
&\quad + \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t)\\
&\le \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} d\mu(t)\\
&\quad + \frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} d\mu(t)\\
&\le \frac{N\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])} + \frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.
\end{aligned}$$

Consequently,

$$\lim_{t\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 d\mu(t) \le \delta\varepsilon, \quad \forall\, \varepsilon > 0.$$

Therefore $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$. □

Next, the proof of Theorem 5.4.

Proof of Theorem 5.4. Assume that $\phi = \phi_1 + \phi_2, h = h_1 + h_2$, where

$$\begin{aligned}
&\phi_1 \in SAP\big(\mathbb{R}\times L^2(P,H_\alpha), L^2(P,H_\alpha)\big), && \phi_2 \in \mathscr{E}\big(\mathbb{R}\times L^2(P,H_\alpha), L^2(P,H_\alpha,\mu,\nu,\infty)\big),\\
&h_1 \in SAP\big(\mathbb{R}, L^2(P,H_\alpha)\big), && h_2 \in \mathscr{E}\big(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,\infty\big).
\end{aligned}$$

Consider

$$\phi\big(t,h(t)\big) = \phi_1\big(t,h_1(t)\big) + \big[\phi\big(t,h(t)\big) - \phi\big(t,h_1(t)\big)\big] + \phi_2\big(t,h_1(t)\big).$$

Using Theorem 5.3, we have $t \mapsto \phi(t,h_1(t)) \in SAP(\mathbb{R}\times L^2(P,H_\alpha), L^2(P,H_\alpha))$. To complete the proof its suffices to show that $t \mapsto [\phi(t,h(t)) - \phi(t,h_1(t))]$ and $t \mapsto \phi_2(t,h_1(t))$ belong to $\mathscr{E}(\mathbb{R}\times L^2(P,H_\alpha), L^2(P,H_\alpha),\mu,\nu,\infty)$. Clearly $t \mapsto [\phi(t,h(t)) - \phi(t,h_1(t))]$ is bounded and continuous. Assume that

$$\mathbb{E}\big\|\phi\big(t,h(t)\big) - \phi\big(t,h_1(t)\big)\big\|_\alpha^2 \le N, \quad \forall\, t \in \mathbb{R}.$$

Since, $h(t)$ and $h_1(t)$ are bounded, we can choose a bounded subset $B \subset \mathbb{R}$ such that $h(\mathbb{R}), h_1(\mathbb{R}) \subset B$. Under assumption (ii), for given $\varepsilon > 0$ such that $\mathbb{E}\|x_1 - x_2\|_\alpha^2 \leq \varepsilon$, implies that

$$\mathbb{E}\|\phi(t, x_1) - \phi(t, x_2)\|_\alpha^2 \leq \varepsilon L_\phi(t), \quad \forall\, t \in \mathbb{R}.$$

Since for $\delta \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$, Lemma 5.1 yields that

$$\lim_{\tau \to +\infty} \frac{\mu\big(M_{\delta,\tau}(\delta)\big)}{\nu([-\tau, \tau])} = 0.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\big\|\phi\big(\theta,h(\theta)\big)-\phi\big(\theta,h_1(\theta)\big)\big\|_\alpha^2\Big)d\mu(t)\\
&=\frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\big\|\phi\big(\theta,h(\theta)\big)-\phi\big(\theta,h_1(\theta)\big)\big\|_\alpha^2\Big)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\big\|\phi\big(\theta,h(\theta)\big)-\phi\big(\theta,h_1(\theta)\big)\big\|_\alpha^2\Big)d\mu(t)\\
&\leq\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\leq\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\leq\frac{N\mu(M_{\tau,\varepsilon}(\delta))}{\nu([-\tau,\tau])}+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t).
\end{aligned}$$

It follows that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\big\|\phi\big(\theta,h(\theta)\big)-\phi\big(\theta,h_1(\theta)\big)\big\|_\alpha^2\Big)d\mu(t)\leq\varepsilon\vartheta,\quad\forall\,\varepsilon>0.$$

Consequently, $t \mapsto [\phi(t, h(t)) - \phi(t, h_1(t))]$ is (μ, ν)-ergodic of infinite class. Now, we will prove that $t \mapsto \phi_2(t, h_1(t))$ is (μ, ν)-ergodic of infinite class. Since ϕ_2 is uniformly continuous on compact set $\Omega = \overline{\{h_1(t) : t \in \mathbb{R}\}}$ which the respect to the second variable x, then for given $\varepsilon > 0$, there exists $\delta > 0$ such that for all ξ_1 and $\xi_2 \in \Omega$, one has $\mathbb{E}\|\xi_1 - \xi_2\|^2 \leq \delta$, implies

$$\mathbb{E}\big\|\phi_2\big(t, \xi_1(t)\big) - \phi_2\big(t, \xi_2(t)\big)\big\|_\alpha^2 \leq \varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset \Omega$ such that

$$\Omega \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i, \delta),$$

and then

$$\mathbb{E}\big\|\phi_2\big(t,h_1(t)\big)\big\|_\alpha^2 \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \mathbb{E}\|\phi_2(t,z_i)\|_\alpha^2.$$

Moreover,

$$\forall i \in \{1,2,\dots,n(\varepsilon)\}, \quad \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|\phi_2(\theta,z_i)\|_\alpha^2 d\mu(t) = 0,$$

then

$$\forall \varepsilon > 0, \quad \limsup_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\big\|\phi_2\big(\theta,h_1(\theta)\big)\big\|_\alpha^2 d\mu(t) \leq \varepsilon,$$

that implies

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\big\|\phi_2\big(\theta,h_1(\theta)\big)\big\|_\alpha^2 d\mu(t) \leq \varepsilon.$$

Consequently $t \mapsto \phi_2(t,h_1(t))$ is (μ,ν)-ergodic of infinite class. □

6 Square-mean (μ,ν)-pseudo almost periodic process of infinite class

Lemma 6.1 (Ito's Isometry, [15])**.** *Let $W : [0,T]\times\Omega \to \mathbb{R}$ denote the canonical real-valued Wiener process defined up to time $T > 0$ and let $X : [0,T]\times\Omega \to \mathbb{R}$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then*

$$\mathbb{E}\Bigg[\bigg(\int_0^T X_t dW_t\bigg)^2\Bigg] = \mathbb{E}\bigg[\int_0^T X_t^2 dt\bigg],$$

where $\mathbb{E}$ denotes expectation with respect to classical Wiener measure.

We make the following assumption.

$(\mathbf{H}_5)$ g is a stochastically bounded process.

Theorem 6.1. *Assume that $(\mathbf{H}_0), (\mathbf{H}_1)$ and $(\mathbf{H}_5)$ hold and the semigroup $(U(t))_{t\geq0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then there exists a unique bounded solution u of Eq.* (1.1) *on $\mathbb{R}$, given by*

$$\begin{aligned} u_t = &\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s) \\ &+ \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s), \end{aligned}$$

where

$$\widetilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_{\mathcal{U}})^{-1}, \quad \lambda > \widetilde{\omega},$$

Π^s *and* Π^u *are projections of* $\mathcal{C}_\alpha$ *onto the stable and unstable subspaces respectively.*

Proof. Let

$$\begin{aligned} u_t = v(t) &+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s) \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s), \end{aligned}$$

where

$$\begin{aligned} v(t) = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds. \end{aligned}$$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [1]. Now we will show that the limit $\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)$ exists. For $t\in\mathbb{R}$ and using the Ito's isometry property of the stochastic integral, we have

$$\begin{aligned} &\mathbb{E}\Big\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big\|_\alpha^2 \\ &\le \mathbb{E}\bigg(\int_{-\infty}^{t}\overline{M}^2\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\big\|\big(\widetilde{B}_\lambda X_0 g(s)\big)\big\|^2 ds\bigg) \\ &\le \overline{M}^2\mathbb{E}\bigg(\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\big\|\big(\widetilde{B}_\lambda X_0 g(s)\big)\big\|^2 ds\bigg) \\ &\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\bigg(\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2 ds\bigg) \\ &\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\mathbb{E}\bigg(\int_{t-n}^{t-n+1}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2 ds\bigg) \\ &\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\bigg[\mathbb{E}\bigg(\int_{t-1}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2 ds\bigg) \\ &\qquad + \sum_{n=2}^{+\infty}\mathbb{E}\bigg(\int_{t-n}^{t-n+1}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2 ds\bigg)\bigg]. \end{aligned}$$

Then, by using the Hölder's inequality, we obtain

$$\mathbb{E}\Big\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big\|_\alpha^2$$

$$
\begin{aligned}
&\leq \overline{M}^2\widetilde{M}^2|\Pi^s|^2\Bigg[\Bigg(\int_{t-1}^{t}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}ds\Bigg)^{\frac{1}{2}}\times\mathbb{E}\Bigg(\int_{t-1}^{t}\|g(s)\|^4ds\Bigg)^{\frac{1}{2}}\\
&\qquad\qquad+\sum_{n=2}^{+\infty}\Bigg(\int_{t-n}^{t-n+1}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}ds\Bigg)^{\frac{1}{2}}\times\mathbb{E}\Bigg(\int_{t-n}^{t-n+1}\|g(s)\|^4ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg[\Bigg(\int_0^{4w}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}+\sum_{n=2}^{+\infty}\Bigg(\int_{4w(n-1)}^{4wn}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg[\Bigg(\int_0^{4w}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\\
&\qquad\qquad\qquad\qquad+\sum_{n=2}^{+\infty}\Bigg(\int_{4w(n-1)}^{4wn}e^{-s}(n-1)^{-4\alpha}(4w)^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg[\Bigg(\int_0^{4w}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\\
&\qquad\qquad\qquad\qquad+\sum_{n=2}^{+\infty}\Bigg((n-1)^{-4\alpha}\int_{4w(n-1)}^{4wn}e^{-s}(4w)^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg(\int_0^{4w}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}\sum_{n=2}^{+\infty}\Bigg(\int_{4w(n-1)}^{4wn}e^{-s}ds\Bigg)^{\frac{1}{2}}\\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg(\int_0^{4w}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(e^{4w}-1)^{\frac{1}{2}}\sum_{n=2}^{+\infty}e^{-2wn}\\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg(\int_0^{4w}e^{-s}s^{-4\alpha}ds\Bigg)^{\frac{1}{2}}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(e^{4w}+1)^{\frac{1}{2}}\sum_{n=2}^{+\infty}e^{-2wn}.
\end{aligned}
$$

Since the series $\sum_{n=2}^{+\infty}e^{-2wn}$ is convergent, it follows that

$$
\mathbb{E}\Bigg\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Bigg\|_\alpha^2 < K, \tag{6.1}
$$

where

$$\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\left(\int_0^{4w}e^{-s}s^{-4\alpha}ds\right)^{\frac{1}{2}}+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(e^{4w}+1)^{\frac{1}{2}}\sum_{n=2}^{+\infty}e^{-2wn}.$$

Set

$$F(n,s,t)=\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big),\quad n\in\mathbb{N},\quad s\le t.$$

For n sufficiently large and $\sigma\le t$ and using the Ito's isometry property of the stochastic integral, we obtain the following result:

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{\sigma}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\right\|_\alpha^2\\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\Bigg[\left(\int_{\sigma-1}^{\sigma}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}ds\right)^{\frac{1}{2}}\times\mathbb{E}\left(\int_{t-1}^{t}\|g(s)\|^4ds\right)^{\frac{1}{2}}\\
&\qquad+\sum_{n=2}^{+\infty}\left(\int_{t-n}^{t-n+1}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}ds\right)^{\frac{1}{2}}\times\mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}\|g(s)\|^4ds\right)^{\frac{1}{2}}\Bigg]\\
&\le\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{(1-4\alpha)/2}}\Bigg[\left(\int_{4w(t-\sigma)}^{4w(t-\sigma+1)}e^{-s}(t-\sigma)^{-4\alpha}(4w)^{-4\alpha}ds\right)^{\frac{1}{2}}\\
&\qquad+\sum_{n=2}^{+\infty}\left(\int_{4w(t-\sigma+n-1)}^{4w(t-\sigma+n)}e^{-s}(t-\sigma+n-1)^{-4\alpha}(4w)^{-4\alpha}ds\right)^{\frac{1}{2}}\Bigg]\\
&\le\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(t-\sigma)^{-2\alpha}\left(\int_{4w(t-\sigma)}^{4w(t-\sigma+1)}e^{-s}ds\right)^{\frac{1}{2}}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(t-\sigma)^{-2\alpha}\left(\int_{4w(t-\sigma+n-1)}^{4w(t-\sigma+n)}e^{-s}ds\right)^{\frac{1}{2}}\\
&\le\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(t-\sigma)^{-2\alpha}(1-e^{-4w})^{\frac{1}{2}}e^{-2w(t-\sigma)}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(e^{4w}+1)^{\frac{1}{2}}e^{-2w(t-\sigma)}\sum_{n=2}^{+\infty}e^{-2wn}\\
&\le K_1(t-\sigma)^{-2\alpha}e^{-2w(t-\sigma)}+K_2e^{-2w(t-\sigma)},
\end{aligned}$$

where

$$K_1=\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(1-e^{-4w})^{\frac{1}{2}},$$
$$K_2=\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{1/2}}(e^{4w}+1)^{\frac{1}{2}}.$$

It follow that for n and m sufficiently large and $\sigma \leq t$, we have

$$
\begin{aligned}
&\mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|_\alpha^2 \\
&\leq \mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(n,s,t)dW(s) + \int_{\sigma}^{t} F(n,s,t)dW(s) \\
&\qquad - \int_{-\infty}^{\sigma} F(m,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|_\alpha^2 \\
&\leq 3\mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big\|_\alpha^2 + 3\mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(m,s,t)dW(s)\Big\|_\alpha^2 \\
&\qquad + 3\mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^2 \\
&\leq 6K_1(t-\sigma)^{-2\alpha}e^{-\omega 2(t-\sigma)} + K_2 e^{-2w(t-\sigma)} \\
&\qquad + 3\mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^2.
\end{aligned}
$$

Since $\lim_{n\to+\infty} \mathbb{E}\| \int_{\sigma}^{t} F(n,s,t)dW(s)\|_\alpha^2$ exists, then

$$
\begin{aligned}
&\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^2 \\
&\leq 6\big(K_1(t-\sigma)^{-2\alpha}e^{-2\omega(t-\sigma)} + K_2 e^{-2w(t-\sigma)}\big).
\end{aligned}
$$

If $\sigma \to -\infty$, then

$$
\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|_\alpha^2 = 0.
$$

We deduce that

$$
\lim_{n\to\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s)\Big\|^2 = \lim_{n\to\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_n X_0 g(s)\big)dW(s)\Big\|_\alpha^2
$$

exists. Therefore, the limit $\lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)$ exists. Moreover, we can see that from the Eq. (6.1) the function

$$
\eta_1 : t \;\to\; \lim_{n\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_n X_0 g(s)\big)dW(s)\Big\|_\alpha^2
$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$
\eta_2 : t \;\to\; \lim_{n\to+\infty} \mathbb{E}\Big\| \int_{t}^{+\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_n X_0 g(s))dW(s)\Big\|_\alpha^2
$$

is well defined and bounded on $\mathbb{R}$. □

Theorem 6.2. *Assume that* $(\mathbf{H}_5)$ *holds. Let* $f, g \in SAP(\mathbb{R}, L^2(P, H_\alpha))$ *and* Γ *be the mapping defined for* $t \in \mathbb{R}$ *by*

$$\begin{aligned}\Gamma(f,g)(t) = \Big[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s) \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big](0).\end{aligned}$$

Then $\Gamma(f,g) \in SAP(\mathbb{R}, L^2(P, H_\alpha))$.

Proof. We can see that $\Gamma(f,g) \in SBC(\mathbb{R}, L^2(P, H_\alpha))$. In fact

$$\begin{aligned}\mathbb{E}\|\Gamma(f,g)(t)\|_\alpha^2 = \mathbb{E}\Big\|\Big[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds \\ &+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s) \\ &+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big](0)\Big\|_\alpha^2 \\ \le 4\mathbb{E}&\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds\Big\|_\alpha^2 \\ + 4\mathbb{E}&\Big\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds\Big\|_\alpha^2 \\ + 4\mathbb{E}&\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big\|_\alpha^2 \\ + 4\mathbb{E}&\Big\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big\|_\alpha^2.\end{aligned}$$

Using Ito's isometry property of stochastic integral, we obtain

$$\begin{aligned}E\|\Gamma(f,g)(t)\|_\alpha^2 \le\ & 4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\|f(s)\|^2ds\right) \\ &+ 4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{t}^{+\infty}\frac{e^{2w(t-s)}}{(s-t)^{2\alpha}}|\Pi^u|^2\|f(s)\|^2ds\right) \\ &+ 4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\|g(s)\|^2ds\right)\end{aligned}$$

$$
\begin{aligned}
&+ 4\mathbb{E}\left(\overline{M}^2 \widetilde{M}^2 \int_t^{+\infty} \frac{e^{2w(t-s)}}{(s-t)^{2\alpha}} |\Pi^u|^2 \|g(s)\|^2 ds\right) \\
&\leq \frac{8\Delta\big(\|f\|_\infty^2 + \|g\|_\infty^2\big)}{(2w)^{1-2\alpha}} \left(\int_0^{+\infty} e^{-s} s^{-2\alpha} ds\right) \\
&= \frac{8\Delta\big(\|f\|_\infty^2 + \|g\|_\infty^2\big)}{(2w)^{1-2\alpha}} \Gamma(1-2\alpha) < \infty, \qquad (6.2)
\end{aligned}
$$

where

$$
\Delta = \max\big(\overline{M}^2 \widetilde{M}^2 |\Pi^s|^2, \overline{M}^2 \widetilde{M}^2 |\Pi^u|^2\big).
$$

Since f and g are square-mean almost periodic process, then the set $\{\Gamma(f,g)_\tau : \tau \in \mathbb{R}\}$ is precompact in $SBC(\mathbb{R}, L^2(P, H_\alpha))$. On other hand, we have

$$
\begin{aligned}
\Gamma(f,g)_\tau(t) &= \Gamma(f,g)(t+\tau) \\
&= \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t+\tau} \mathcal{U}^s(t+\tau-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big) ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{+\infty}^{t+\tau} \mathcal{U}^u(t+\tau-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big) ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{-\infty}^{t+\tau} \mathcal{U}^s(t+\tau-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big) dW(s) \\
&\quad + \lim_{\lambda\to+\infty} \int_{+\infty}^{t+\tau} \mathcal{U}^u(t+\tau-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big) dW(s)\Big] \\
&= \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(\tau+s)\big) ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(\tau+s)\big) ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(\tau+s)\big) dW(s) \\
&\quad + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(\tau+s)\big) dW(s)\Big] \\
&= \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f_\tau(s)\big) ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f_\tau(s)\big) ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g_\tau(s)\big) dW(s) \\
&\quad + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g_\tau(s)\big) dW(s)\Big] \\
&= \Gamma(f_\tau, g_\tau)(t), \quad \forall\, t \in \mathbb{R}.
\end{aligned}
$$

Thus, $\Gamma(f,g)_\tau = \Gamma(f_\tau, g_\tau)$ which implies $\{\Gamma(f,g)_\tau : \tau \in \mathbb{R}\}$ is relatively compact in $SBC(\mathbb{R}, L^2(P,H_\alpha))$. Since Γ is continuous $SBC(\mathbb{R}, L^2(P,H_\alpha))$ into $SBC(\mathbb{R}, L^2(P,H_\alpha))$, then $\Gamma(f,g) \in SAP(\mathbb{R}, L^2(P,H_\alpha))$. $\square$

Theorem 6.3. *Assume that* $(\mathbf{H}_2)$ *and* $(\mathbf{H}_5)$ *hold. Let* $f,g \in \mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,\infty)$*, then* $\Gamma(f,g) \in \mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,\infty)$.

Proof.

$$\begin{aligned}
\Gamma(f,g)(t) = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\|\Gamma(f,g)(t)\|_\alpha^2 = \mathbb{E}\Big\| &\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s)\big)ds\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 f(s)\big)ds\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u\big(\widetilde{B}_\lambda X_0 g(s)\big)dW(s)\Big\|_\alpha^2,
\end{aligned}$$

then by Ito's isometry property of stochastic integral, we have

$$\begin{aligned}
&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|\Gamma(f,g)(\theta)\|_\alpha^2\Big)d\mu(t)\\
&\le \int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\Big[4\overline{M}^2\widetilde{M}^2\mathbb{E}\Big(\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\|f(s)\|^2ds\\
&\qquad +\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\|f(s)\|^2ds\\
&\qquad +\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\|g(s)\|^2ds\\
&\qquad +\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\|g(s)\|^2ds\Big)\Big]d\mu(t)\\
&\le 4\overline{M}^2\widetilde{M}^2\Big[\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|f(s)\|^2ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\mathbb{E}\|f(s)\|^2ds\\
&+\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|g(s)\|^2ds\\
&+\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\mathbb{E}\|g(s)\|^2ds\Big)\Big]d\mu(t)\\
\leq \Delta\Big[&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}\big(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2\big)ds\Big)d\mu(t)\\
&+\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}\big(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2\big)ds\Big)d\mu(t)\Big],
\end{aligned}
$$

where

$$\Delta=\max\big(4\overline{M}^2\widetilde{M}^2|\Pi^s|^2,4\overline{M}^2\widetilde{M}^2|\Pi^u|^2\big).$$

On one hand by Fubini's theorem, we have

$$
\begin{aligned}
&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}\big(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2\big)ds\Big)d\mu(t)\\
\leq&\int_{-\tau}^{\tau}\Big(\int_{0}^{+\infty}\sup_{\theta\in]-\infty,t]}\frac{e^{-2ws}}{s^{2\alpha}}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)ds\Big)d\mu(t)\\
\leq&\int_{0}^{+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)d\mu(t)\Big)ds.
\end{aligned}
$$

Moreover, by Lebesgue dominated convergence theorem and the Theorem 4.3, we deduce that

$$\lim_{\tau\to+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)ds\Big)d\mu(t)=0$$

for all $s\in\mathbb{R}^+$, and

$$
\begin{aligned}
&\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)ds\Big)d\mu(t)\\
\leq&\frac{e^{-2ws}}{s^{2\alpha}}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\big(\|f\|_\infty^2+\|g\|_\infty^2\big).
\end{aligned}
$$

Since f and g are bounded functions that

$$s\ \mapsto\ \frac{e^{-2ws}}{s^{2\alpha}}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\big(\|f\|_\infty^2+\|g\|_\infty^2\big)$$

belongs to $L^1([0,+\infty])$ in view of the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\left(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\left(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\right)d\mu(t)\right)ds=0.$$

Since like above, it follows that

$$\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\left(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\left(\mathbb{E}\|f(\theta+s)\|^2+\mathbb{E}\|g(\theta+s)\|^2\right)d\mu(t)\right)ds=0.$$

Consequently,

$$\lim_{\tau\to+\infty}\frac{1}{\nu([(-\tau,\tau)])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|\Gamma(f,g)(\theta)\|_\alpha^2\right)d\mu(t)=0.$$

Thus, we obtain the desired result. □

For proof of existence of square-mean compact pseudo almost periodic solution of infinite class, we need the following assertion.

$(\mathbf{H}_6)$ $f,g:\mathbb{R}\to L^2(P,H)$ are compact α-$cl(\mu,\nu)$-pseudo almost periodic of infinite class.

Theorem 6.4. *Assume that* $(\mathbf{H}_0),(\mathbf{H}_1),(\mathbf{H}_3)$ *and* $(\mathbf{H}_6)$ *hold. Then Eq.* (1.1) *has a unique compact* α-$cl(\mu,\nu)$ *pseudo almost periodic solution of infinite class.*

Proof. Since f and g are (μ,ν)- pseudo almost periodic functions, f has a decomposition $f=f_1+f_2$ and $g=g_1+g_2$, where $f_1,g_1\in SAP(\mathbb{R},L^2(P,H_\alpha))$ and $f_2,g_2\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,\infty)$. Using Theorems 6.1, 6.2 and 6.3, we get the desired result. □

Our next objective is to show the existence of square mean α-(μ,ν)-pseudo almost periodic solutions of infinite class for the following problem:

$$dx(t)=[-Ax(t)+L(x_t)+f(t,u_t)]dt+g(t,u_t)dW(t),\quad t\in\mathbb{R},\tag{6.3}$$

where $f,g:\mathbb{R}\times\mathcal{C}_\alpha\to L^2(P,H)$ are two continuous stochastic processes.

For the sequel, we formulate the following assumptions.

$(\mathbf{H}_7)$ Let $\mu,\nu\in\mathcal{M}$ and $f:\mathbb{R}\times\mathcal{B}_\alpha\to L^2(P,H_\alpha)$ be a square mean $cl(\mu,\nu)$-pseudo almost periodic of infinite class such that there exists a positive constant L_f such that

$$\mathbb{E}\|f(t,\phi_1)-f(t,\phi_2)\|^2\le L_f\mathbb{E}\|\phi_1-\phi_2\|_\alpha^2,\quad\forall\, t\in\mathbb{R},\quad \phi_1,\phi_2\in\mathcal{B}_\alpha.$$

$(\mathbf{H}_8)$ Let $\mu,\nu\in\mathcal{M}$ and $g:\mathbb{R}\times\mathcal{B}_\alpha\to L^2(P,H_\alpha)$ be a square mean $cl(\mu,\nu)$ pseudo almost periodic of class r such that there exists a positive constant L_g such that

$$\mathbb{E}\|g(t,\phi_1)-g(t,\phi_2)\|^2\le L_g\mathbb{E}\|\phi_1-\phi_2\|_\alpha^2,\quad\forall\, t\in\mathbb{R},\quad \phi_1,\phi_2\in\mathcal{B}_\alpha.$$

$(\mathbf{H}_9)$ The instable space $U\equiv\{0\}$.

Theorem 6.5. *Assume that $\mathcal{B}_\alpha$ satisfies $(\mathbf{A}_1), (\mathbf{A}_2), (\mathbf{B}), (\mathbf{C}_1), (\mathbf{C}_2)$, and $(\mathbf{C}_3)$ and $(\mathbf{H}_0)$, $(\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3), (\mathbf{H}_4), (\mathbf{H}_7), (\mathbf{H}_8)$ and $(\mathbf{H}_9)$ hold. If*

$$\frac{2\overline{M}\widetilde{M}|\Pi^s|\sqrt{2\eta(|L_f+L_g)}}{(2w)^{(1-2\alpha)/2}}\sqrt{\Gamma(1-2\alpha)}<1,$$

then Eq. (6.3) *has unique α-cl(μ,ν)-square-mean pseudo almost periodic solution of infinite class.*

Proof. Let x be a function in $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$. From Theorem 5.2, the function $t \to x_t$ belongs to $SPAP(\mathcal{B}_\alpha, \mu, \nu, \infty)$. Hence Theorem 5.4 implies that the function $g(\cdot) = f(\cdot, x)$ is in $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$. Since by $(\mathbf{H}_9)$, the instable space $U \equiv \{0\}$, then $|\Pi^u| = 0$. Moreover $\mathcal{B}_\alpha$ is fading memory space, by Lemma 3.3, we can choose the function K and M such that $M(t) \to 0$ as $t \to +\infty$. Let

$$\eta = \max_{t\in\mathbb{R}}\left\{\sup_{t\in\mathbb{R}}|K(t)|^2, \sup_{t\in\mathbb{R}}|M(t)|^2\right\},$$

and consider the mapping

$$\mathcal{H} : SPAP\big(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty\big) \;\to\; SAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 f(s,x_s)\big)ds + \lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0 g(s,x_s)\big)dW(s)\Big](0).$$

From Theorems 6.1-6.3, it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$. Let $x_1, x_2 \in SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$, then we have

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{H}x_1)-(\mathcal{H}x_2)\|_\alpha^2\\
&\le 2\mathbb{E}\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0\big(f(s,x_{1s})-f(s,x_{2s})\big)\big)ds\Big\|_\alpha^2\\
&\quad + 2\mathbb{E}\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0\big(g(s,x_{1s})-g(s,x_{2s})\big)\big)dW(s)\Big\|_\alpha^2.
\end{aligned}$$

By Ito's isometry property, it follows that

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)\|_\alpha^2\\
&\le 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}L_f\mathbb{E}\|x_{1s}-x_{2s}\|_{\mathcal{B}_\alpha}^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_g \mathbb{E}\|x_{1s}-x_{2s}\|^2_{\mathcal{B}_\alpha} ds \\
&\leq 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_f \mathbb{E}\Big(K(s)\sup_{0\leq\xi\leq s}\|_\alpha x_1(\xi)-x_2(\xi)\|_\alpha \\
&\qquad\qquad + M(s)\|x_{1_0}-x_{2_0}\|_\alpha\Big)^2 ds \\
&+ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_g \Big(K(s)\sup_{0\leq\xi\leq s}\|x_1(\xi)-x_2(\xi)\|_\alpha \\
&\qquad\qquad + M(s)\|x_{1_0}-x_{2_0}\|_\alpha\Big)^2 ds \\
&\leq 4\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_f \mathbb{E}\Big(K^2(s)\sup_{0\leq\xi\leq s}\|x_1(\xi)-x_2(\xi)\|^2_\alpha \\
&\qquad\qquad + M^2(s)\|x_{1_0}-x_{2_0}\|_\alpha\Big) ds \\
&+ 4\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_g \mathbb{E}\Big(K^2(s)\sup_{0\leq\xi\leq s}\|x_1(\xi)-x_2(\xi)\|^2_\alpha \\
&\qquad\qquad + M^2(s)\|x_{1_0}-x_{2_0}\|_\alpha\Big) ds \\
&\leq \frac{8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta(L_f+L_g)}{(2w)^{1-2\alpha}}\left(\int_0^{+\infty} e^{-s}s^{-2\alpha}ds\right)\|x_1-x_2\|^2_{\infty,\alpha} \\
&\leq \frac{8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta(|L_f+L_g)}{(2w)^{1-2\alpha}}\Gamma(1-2\alpha)\|x_1-x_2\|^2_{\infty,\alpha}.
\end{aligned}
$$

Consequently,

$$
\|(\mathcal{H}x_1)-(\mathcal{H}x_1)\|_{\infty,\alpha} \leq \frac{2\overline{M}\widetilde{M}|\Pi^s|\sqrt{2\eta(L_f+L_g)}}{(2w)^{(1-2\alpha)/2}}\sqrt{\Gamma(1-2\alpha)}\|x_1-x_2\|_{\infty,\alpha}.
$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $SPAP(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, \infty)$. We conclude that Eq. (6.3), has one and only one α-$cl(\mu,\nu)$-square-mean pseudo almost periodic solution of infinite class. The proof is complete. □

Proposition 6.1. *Assume that* $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3), (\mathbf{H}_4), (\mathbf{H}_9)$ *hold and* f, g *are Lipschitz continuous with respect the second argument. If*

$$
\mathrm{Lip}(f) = \mathrm{Lip}(g) < \frac{(2w)^{1-2\alpha}}{16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\Gamma(1-2\alpha)},
$$

then Eq. (6.3) *has unique α-$cl(\mu,\nu)$-square-mean pseudo almost periodic solution of infinite class, where* $\mathrm{Lip}(f)$ *and* $\mathrm{Lip}(g)$ *are respectively the Lipschitz constant of f and g.*

Proof. Let us pose $k = \mathrm{Lip}(f) = \mathrm{Lip}(g)$. By Ito's isometry property, we have

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|_\alpha^2 \\
&\leq 4\overline{M}^2\widetilde{M}^2|\Pi^s|^2 \int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_f \mathbb{E}\Big(\|K^2(s) \sup_{0\leq\xi\leq s} \|x_1(\xi) - x_2(\xi)\|_\alpha^2 \\
&\qquad + M^2(s)\|x_{1_0} - x_{2_0}\|_\alpha^2\Big) ds \\
&\quad + 4\overline{M}^2\widetilde{M}^2|\Pi^s|^2 \int_{-\infty}^t \frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}} L_g \mathbb{E}\Big(\|K^2(s) \sup_{0\leq\xi\leq s} \|x_1(\xi) - x_2(\xi)\|_\alpha^2 \\
&\qquad + M^2(s)\|x_{1_0} - x_{2_0}\|_\alpha^2\Big) ds \\
&\leq \frac{16\overline{M}^2\widetilde{M}^2|\Pi^s|^2 k\eta}{(2w)^{1-2\alpha}} \left(\int_0^{+\infty} e^{-s}s^{-2\alpha} ds\right)\|x_1 - x_2\|_{\infty,\alpha}^2 \\
&\leq \frac{16\overline{M}^2\widetilde{M}^2|\Pi^s|^2 k\eta}{(2w)^{1-2\alpha}} \Gamma(1-2\alpha)\|x_1 - x_2\|_{\infty,\alpha}^2.
\end{aligned}$$

Consequently,

$$\|(\mathcal{H}x_1) - (\mathcal{H}x_1)\|_{\infty,\alpha} \leq \frac{4\overline{M}\widetilde{M}|\Pi^s|\sqrt{k\eta}}{(2w)^{(1-2\alpha)/2}} \sqrt{\Gamma(1-2\alpha)}\|x_1 - x_2\|_{\infty,\alpha}.$$

This means $\mathcal{H}$ is strict contraction if

$$k < \frac{(2w)^{1-2\alpha}}{16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Gamma(1-2\alpha)}.$$

The proof is complete. □

7 Application

For illustration, we propose to study the existence of solutions for the following model:

$$\begin{cases}
dz(t,x) = -\dfrac{\partial^2}{\partial x^2}z(t,x)dt + \Big[\displaystyle\int_{-\infty}^0 G(\theta)z(t+\theta,x)d\theta + x\big(\sin(t) + \sin(\sqrt{2}t)\big) \\
\qquad + \arctan(t) + \displaystyle\int_{-\infty}^0 h\Big(\theta, \frac{\partial}{\partial x}z(t+\theta,x)\Big)d\theta\Big] dt \\
\qquad + \Big[x\dfrac{\cos(t)}{2+\cos(\sqrt{2}t)} + \cos(t) + \displaystyle\int_{-\infty}^0 h\Big(\theta, \frac{\partial}{\partial x}z(t+\theta,x)\Big)d\theta\Big] dW(t), \\
t \in \mathbb{R}, \quad x \in [0,\pi], \\
z(t,0) = z(t,\pi) = 0, \quad t \in \mathbb{R}, \quad x \in [0,\pi],
\end{cases} \tag{7.1}$$

where $G : \mathbb{R}_- \longrightarrow \mathbb{R}$ is continuous function and $h : \mathbb{R}_- \to \mathbb{R}$ is Lipschitz continuous with the respect of the second argument. For example, we can take

$$G(\theta) = e^{\gamma(\theta+1)}, \qquad \gamma > 0, \quad \theta \in]-\infty, 0],$$
$$h(\theta, x) = \theta^2 + \sin\left(\frac{x}{4}\right), \quad (\theta, x) \in]-\infty, 0] \times \mathbb{R}.$$

We can see that G is continuous and

$$|h(\theta, x_1) - h(\theta, x_2)| \leq \frac{1}{4}|x_1 - x_2|,$$

which implies that h is lipschitz continuous with the respect of the second argument. $W(t)$ is a two-sided standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with

$$\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u, v \leq t\}.$$

Let $\gamma > 0$, we define the phase space

$$\mathcal{B} = C_\gamma = \left\{\varphi \in C\big(]-\infty, 0]; L^2(P, H)\big) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } L^2(P, H)\right\}$$

with the norm

$$\|\varphi\|_\gamma^2 = \sup_{\theta \leq 0} \mathbb{E}\|e^{\gamma\theta}\varphi(\theta)\|^2, \quad \varphi \in C_\gamma.$$

From [13], this space satisfies axioms $(\mathbf{A}), (\mathbf{B}), (\mathbf{C})$ and $(\mathbf{D})$. Moreover it is a uniform fading memory space, which implies that $(\mathbf{H}_3)$ is satisfied. We choose $\alpha = 1/2$. The norm in $\mathcal{B}_{1/2}$ is given by

$$\|\varphi\|_{\frac{1}{2}}^2 = \sup_{\theta \leq 0} \mathbb{E}\left\|e^{\gamma\theta}\big(A^{\frac{1}{2}}\varphi(\theta)\big)\right\|^2 = \sup_{\theta \leq 0}\left(\int_0^\pi \mathbb{E}\left|e^{\gamma\theta}\left(\frac{\partial}{\partial x}(\varphi)(\theta)(x)\right)\right|^2 dx\right).$$

Moreover, $A^{-1/2}\varphi \in \mathcal{B}$, for $\varphi \in \mathcal{B}$, then $(\mathbf{H}_1)$ is satisfied.

To rewrite Eq. (7.1) in abstract form, we introduce the space $H = L^2((0, \pi))$. Let $A : D(A) \to L^2((0, \pi))$ defined by

$$\begin{cases} D(A) = H^2(0, \pi) \cap H^1(0, \pi), \\ Ay(t) = y''(t), \quad t \in (0, \pi), \quad y \in D(A). \end{cases}$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 : \; n \geq 1\},$$

and the associated eigenfunctions $(\zeta_n)_{n \geq 1}$ are given by

$$\zeta_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns), \quad s \in [0, \pi].$$

Then the operator A is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2 (y, \zeta_n)\zeta_n, \quad y \in D(A).$$

For each

$$y \in D(A^{\frac{1}{2}}) = \left\{ y \in H : \sum_{n=1}^{+\infty} n(y, \zeta_n)\zeta_n \in H \right\},$$

we define the fractional power

$$A^{\frac{1}{2}} : D(A^{\frac{1}{2}}) \subset H \rightarrow H$$

by

$$A^{\frac{1}{2}} y = \sum_{n=1}^{+\infty} n(y, \zeta_n)\zeta_n, \quad y \in D(A^{\frac{1}{2}}).$$

It is well known that $-A$ is the generator of a compact analytic semigroup $(T(t))_{t\geq 0}$ on $L^2((0,\pi))$ which is given by

$$T(t)u = \sum_{n=1}^{+\infty} e^{-n^2 t}(u, \zeta_n)\zeta_n, \quad u \in L^2\big((0,\pi)\big).$$

Then $(\mathbf{H}_0)$ and $(\mathbf{H}_2)$ are satisfied.

Now, we define $f : \mathbb{R} \times \mathcal{B}_{1/2} \to L^2((0,\pi))$ and $L : \mathcal{B}_{1/2} \to L^2(0,\pi)$ as follows:

$$\begin{aligned} f(t,\phi)(x) &= x\big(\sin(t) + \sin(\sqrt{2}t)\big) + \arctan(t) \\ &\quad + \int_{-\infty}^{0} h\Big(\theta, \frac{\partial}{\partial x}\phi(\theta)(x)\Big) d\theta, \quad x \in (0,\pi), \quad t \in \mathbb{R}, \\ g(t,\phi)(x) &= x\frac{\cos(t)}{2+\cos(\sqrt{2}t)} + \cos(t) \\ &\quad + \int_{-\infty}^{0} h\Big(\theta, \frac{\partial}{\partial x}\phi(\theta)(x)\Big) d\theta, \quad x \in (0,\pi), \quad t \in \mathbb{R}, \\ L(\phi)(x) &= \int_{-r}^{0} G(\theta)\phi(\theta)(x), \qquad -r \leq \theta, \qquad x \in (0,\pi). \end{aligned}$$

Lemma 7.1 ([16]). *If $y \in D(A^{1/2})$, then y is absolutely continuous, $y' \in L^2(P,H)$ and $|y'| = |A^{1/2}y|$.*

Let us pose $v(t)(x) = z(t,x)$. Then Eq. (7.1) takes the following abstract form:

$$dv(t) = [-Av(t) + L(v_t) + f(t,v_t)]dt + g(t,v_t)dW(t), \quad t \in \mathbb{R}. \tag{7.2}$$

Consider the measure μ and ν where its Randon-Nikodym derivates are respectively ρ_1 and ρ_2

$$\rho_1(t) = \begin{cases} 1, & t > 0, \\ e^t, & t \leq 0, \end{cases}$$

and

$$\rho_2(t) = |t|, \quad t \in \mathbb{R},$$

i.e. $d\mu(t) = \rho_1(t)dt$ and $d\mu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \quad \text{for} \quad \nu(A) = \rho_2(t)dt, \quad A \in \mathcal{N}.$$

From [7], $\mu, \nu \in \mathcal{M}$ satisfy hypothesis $(\mathbf{H}_4)$. We have also

$$\lim_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\int_{-r}^{0} e^t dt + \int_0^\tau dt}{2\int_0^\tau t dt} = \limsup_{\tau\to+\infty} \frac{1+e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H}_2)$ is satisfied. On other hand, one can see that

$$A^{\frac{1}{2}}\big(x\big(\sin(t)+\sin(\sqrt{2}t)\big)\big) = \big(x\big(\sin(t)+\sin(\sqrt{2}t)\big)\big)' = \sin(t)+\sin(\sqrt{2}t),$$

and $t \mapsto \sin(t)+\sin(\sqrt{2}t)$ is almost periodic. Then $t \longmapsto x(\sin(t)+\sin(\sqrt{2}t))$ is α-almost periodic.

By Lemma 7.1, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}|\arctan(\theta)|^2_{\frac{1}{2}} d\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\big|A^{\frac{1}{2}}\arctan(\theta)\big|^2 d\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\left|\frac{1}{1+\theta^2}\right|^2 d\mu(t) \\
&\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} d\mu(t) \\
&\le \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \;\to\; 0 \quad \text{as} \quad \tau \;\to\; +\infty.
\end{aligned}$$

Thus, $t \mapsto x(\sin(t)+\sin(\sqrt{2}t)) + \arctan(t)$ belongs to $SPAP(\mathbb{R}, L^2(P, H_{1/2}), \mu, \nu, \infty)$. Moreover, for every $\phi_1, \phi_2 \in \mathcal{B}_{1/2}$, by Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{E}\|f(t,\phi_1) - f(t,\phi_2)\|^2_{L^2} \\
&= \mathbb{E}\left[\int_0^\pi \left(\int_{-\infty}^0 h\Big(t, \frac{\partial}{\partial x}\phi_1(\theta)(x)\Big) - h\Big(t, \frac{\partial}{\partial x}\phi_2(\theta)(x)\Big) d\theta\right)^2 dx\right] \\
&\le \mathbb{E}\left[\int_0^\pi \int_{-\infty}^0 h_1^2(\theta)\left|\frac{\partial}{\partial x}\phi_1(\theta)(x) - \frac{\partial}{\partial x}\phi_2(\theta)(x)\right|^2 d\theta dx\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\Big[\int_0^{\pi}\int_{-\infty}^{0} h_1^2(\theta)e^{-2\gamma\theta}e^{2\gamma\theta}\Big|\frac{\partial}{\partial x}\phi_1(\theta)(x)-\frac{\partial}{\partial x}\phi_2(\theta)(x)\Big|^2 d\theta dx\Big]\\
&\leq \mathbb{E}\Big[\int_0^{\pi}\Big(\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h_1^2(\theta)d\theta\Big)\Big(\int_{-\infty}^{0} e^{4\gamma\theta}\Big|\frac{\partial}{\partial x}\phi_1(\theta)(x)-\frac{\partial}{\partial x}\phi_2(\theta)(x)\Big|^2 d\theta\Big)\Big)dx\Big]\\
&\leq \Big[\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h_1^2(\theta)d\theta\Big)\int_0^{\pi}\Big(\int_{-\infty}^{0} e^{4\gamma\theta}\mathbb{E}\Big|\frac{\partial}{\partial x}\phi_1(\theta)(x)-\frac{\partial}{\partial x}\phi_2(\theta)(x)\Big|^2 d\theta\Big)\Big]\\
&\leq \Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h_1^2(\theta)d\theta\Big)\Big(\int_{-\infty}^{0} e^{2\gamma\theta}\int_0^{\pi}\mathbb{E}\Big|e^{\gamma\theta}\Big(\frac{\partial}{\partial x}\phi_1(\theta)(x)-\frac{\partial}{\partial x}\phi_2(\theta)(x)\Big)\Big|^2 d\theta\Big)dx\\
&\leq \Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h_1^2(\theta)d\theta\Big)\sup_{\theta\leq 0}\int_0^{\pi}\mathbb{E}\Big|e^{\gamma\theta}\Big(\frac{\partial}{\partial x}\phi_1(\theta)(x)-\frac{\partial}{\partial x}\phi_2(\theta)(x)\Big)\Big|^2 dx\Big)\Big(\int_{-\infty}^{0} e^{2\gamma\theta}d\theta\Big)\\
&\leq \frac{1}{2\gamma}\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h_1^2(\theta)d\theta\Big)\sup_{\theta\leq 0}\int_0^{\pi}\mathbb{E}\Big|e^{\gamma\theta}\Big(\frac{\partial}{\partial x}\phi_1(\theta)(x)-\frac{\partial}{\partial x}\phi_2(\theta)(x)\Big)\Big|^2 dx\Big)\\
&\leq \frac{1}{2\gamma}\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h_1^2(\theta)d\theta\Big)\|\phi_1-\phi_2\|^2_{\frac{1}{2}}.
\end{aligned}
$$

This means f is Lipchitz continuous. In addition

$$
\begin{aligned}
\mathbb{E}\|f(t,\phi)\|^2_{L^2} &\leq 3\frac{\pi^2}{2}+3\frac{\pi^2}{2}+3\mathbb{E}\Big[\int_0^{\pi}\Big(\int_{-\infty}^{0} h\Big(t,\frac{\partial}{\partial x}\phi(\theta)(x)\Big)|d\theta\Big)^2 dx\Big]\\
&\leq 3\pi^2+3\mathbb{E}\Big[\int_0^{\pi}\Big(\int_{-\infty}^{0} h_1^2(\theta)e^{-2\gamma\theta}e^{2\gamma\theta}\Big|\frac{\partial}{\partial x}\phi(\theta)(x)\Big|^2 d\theta\Big)dx\Big]\\
&\leq 3\pi^2+3\int_{-\infty}^{0} h_1^2(\theta)e^{-2\gamma\theta}\Big(\sup_{\theta\leq 0}\int_0^{\pi}\mathbb{E}\Big|e^{\gamma\theta}\Big(\frac{\partial}{\partial x}\phi(\theta)(x)\Big)\Big|^2 dx\Big)d\theta\\
&\leq 3\pi^2+3\Big(\int_{-\infty}^{0} h_1^2(\theta)e^{-2\gamma\theta}d\theta\Big)\|\phi\|^2_{\frac{1}{2}}<\infty.
\end{aligned}
$$

This means that f is bounded. Consequently, we conclude that f Lipschitz continuous and square mean α-$cl(\mu,\nu)$ pseudo almost periodic of infinite class. Then $(\mathbf{H}_7)$ yields. Similarly, by the same

$$
A^{\frac{1}{2}}\Big(x\frac{\cos(t)}{2+\cos(\sqrt{2}t)}\Big)=\Big(x\frac{\cos(t)}{2+\cos(\sqrt{2}t)}\Big)'=\frac{\cos(t)}{2+\cos(\sqrt{2}t)}.
$$

Then

$$
t \longmapsto x\frac{\cos(t)}{2+\cos(\sqrt{2}t)}
$$

is α-almost periodic. Since for $\theta\in\mathbb{R}$, $-1\leq\sin\theta\leq 1$ and by Lemma 7.1, we have

$$
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}|\cos(\theta)|^2_{\frac{1}{2}}d\mu(t)
$$

$$
\begin{aligned}
&= \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}\big|A^{\frac{1}{2}}\cos(\theta)\big|^2 d\mu(t)\\
&= \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}|\sin(\theta)|^2 d\mu(t)\\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} d\mu(t)\\
&\leq \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \quad\to\quad 0 \quad \text{as} \quad \tau\to+\infty.
\end{aligned}
$$

This means that $t\to\cos(t)$ is α-(μ,ν)-ergodic of infinite class. Moreover,

$$
\begin{aligned}
\mathbb{E}\|g(t,\phi)\|_{L^2}^2 &\leq 3\frac{\pi^2}{2}+3\frac{\pi^2}{2}+3\mathbb{E}\Big[\int_0^{\pi}\Big(\int_{-\infty}^0 h\Big(t,\frac{\partial}{\partial x}\phi(\theta)(x)\Big)d\theta\Big)^2 dx\Big]\\
&\leq 3\pi^2+3\mathbb{E}\Big[\int_0^{\pi}\Big(\int_{-\infty}^0 h_1^2(\theta)e^{-2\gamma\theta}e^{2\gamma\theta}\Big|\frac{\partial}{\partial x}\phi(\theta)(x)\Big|^2 d\theta\Big)dx\Big]\\
&\leq 3\pi^2+3\int_{-\infty}^0 h_1^2(\theta)e^{-2\gamma\theta}\Big(\sup_{\theta\leq 0}\int_0^{\pi}\mathbb{E}\Big|e^{\gamma\theta}\Big(\frac{\partial}{\partial x}\phi(\theta)(x)\Big)\Big|^2 dx\Big)d\theta\\
&\leq 3\pi^2+3\Big(\int_{-\infty}^0 h_1^2(\theta)e^{-2\gamma\theta}d\theta\Big)\|\phi\|_{\frac{1}{2}}^2<\infty.
\end{aligned}
$$

Then hypothesis $(\mathbf{H}_5)$ is verified. Moreover, by Ito's isometry property, Hölder's equality and Fubini's theorem, we have

$$
\|g(t,\phi_1)-g(t,\phi_2)\|^2\leq\frac{1}{2\gamma}\Big(\int_{-\infty}^0 e^{-2\gamma\theta}h^2(\theta)d\theta\Big)\|\phi_1-\phi_2\|_{\frac{1}{2}}^2.
$$

Consequently, we conclude that g Lipschitz continuous and square mean α-$cl(\mu,\nu)$ pseudo almost periodic of infinite class. Then $(\mathbf{H}_8)$ yields. For hyperbolycity , we suppose that

$(\mathbf{H}_{10})$ $\displaystyle\int_{-r}^0|G(\theta)|d\theta<1.$

Lemma 7.2 ([12]). *Assume that $(\mathbf{H}_{10})$ holds. Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic and the instable space $U\equiv\{0\}$.*

We can in our case see that

$$
\int_{-\infty}^0|G(\theta)|d\theta=\lim_{r\to+\infty}\int_{-r}^0 e^{(\gamma+1)\theta}d\theta=\lim_{r\to+\infty}\Big[\frac{1}{\gamma+1}e^{(\gamma+1)\theta}\Big]_{-r}^0=\frac{1}{\gamma+1}<1.
$$

Theorem 7.1. *Assume that $(\mathbf{H}_7),(\mathbf{H}_8),(\mathbf{H}_9)$ and $(\mathbf{H}_{10})$ hold. If* $\mathrm{Lip}(h)$ *is small enough, then Eq.* (7.2) *has a unique α-$cl(\mu,\nu)$-square-mean pseudo almost periodic solution v of infinite class.*

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