Analysis in Theory and Applications
Anal. Theory Appl., Vol. **35**, No. 4 (2019), pp. 377-391

DOI: 10.4208/ata.OA-2018-1012

Weighted Norm Inequalities for Toeplitz Type Operator Related to Singular Integral Operator with Variable Kernel

Yuexiang He*

Department of Mathematics, Jiaozuo University, Jiaozuo 454003, Henan, China

Received 21 November 2018; Accepted (in revised version) 29 September 2019

Abstract. Let $T^{k,1}$ be the singular integrals with variable Calderón-Zygmund kernels or $\pm I$ (the identity operator), let $T^{k,2}$ and $T^{k,4}$ be the linear operators, and let $T^{k,3} = \pm I$. Denote the Toeplitz type operator by

$$T^b = \sum_{k=1}^{t} (T^{k,1} M^b I_\alpha T^{k,2} + T^{k,3} I_\alpha M^b T^{k,4}),$$

where $M^b f = bf$, and I_α is the fractional integral operator. In this paper, we investigate the boundedness of the operator on weighted Lebesgue space when b belongs to weighted Lipschitz space.

Key Words: Toeplitz type operator, variable Calderón-Zygmund kernel, fractional integral, weighted Lipschitz space.

AMS Subject Classifications: 42B20, 42B25

1 Introduction and results

As the development of the singular integral operators, their commutators have been well studied (see [1-3]). In [1], the authors proved that the commutators $[b, T]$, which generated by Calderón-Zygmund singular integral operators and BMO functions, are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo [4] obtained a similar result when Calderón-Zygmund singular integral operators are replaced by the fractional integral operators. Recently, some Toeplitz type operators related to the singular integral operators are introduced, and the boundedness for the operators generated by singular integral operators and BMO functions or Lipschitz functions are obtained (see [5-8]).

*Corresponding author. *Email address:* `heyuexiang63@163.com` (Y. X. He)

Let $K(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a variable Calderón-Zygmund kernel, which depends on some parameter x and possesses 'good' properties with respect to the second variable ξ. The singular integral operator with variable Calderón-Zygmund kernel is defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x, x-y) f(y) dy. \tag{1.1}$$

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator associated to singular integrals with variable Calderón-Zygmund kernel and fractional integral operator I_α is defined by

$$T^b = \sum_{k=1}^{t} (T^{k,1} M^b I_\alpha T^{k,2} + T^{k,3} I_\alpha M^b T^{k,4}), \tag{1.2}$$

where $M^b f = bf$, and $T^{k,1}$ are the singular integral operators with variable Calderón-Zygmund kernel or $\pm I$ (the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3} = \pm I, k = 1, \cdots, t$.

Note that the commutators $[b, I_\alpha](f) = bI_\alpha(f) - I_\alpha(bf)$ are the particular operators of the Toeplitz type operators T^b. The Toeplitz type operators T^b are the non-trivial generalizations of these commutators.

It is well known that the commutators of fractional integral have been widely studied by many authors. Paluszyński [9] showed that $b \in Lip_\beta$ (homogeneous Lipschitz space) if and only if $[b, I_\alpha]$ is bounded from L^p to L^q, where $0 < \beta < 1, 1 < p < n/(\alpha+\beta)$ and $1/q = 1/p - (\alpha+\beta)/n$. When b belongs to the weighted Lipschitz spaces $Lip_{\beta,\omega}$, Hu and Gu [10] proved that $[b, I_\alpha]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$ for $1/q = 1/p - (\alpha+\beta)/n$ with $1 < p < n/(\alpha+\beta)$. A similar result obtained when I_α is replaced by the generalized fractional integral operator [7].

Motivated by these papers, in this paper, we investigate the boundedness of the Toeplitz type operator as (1.2) on weighted Lebesgue space when b belongs to weighted Lipschitz space, and get the following result.

Theorem 1.1. *Suppose that T is a singular integral operator with variable Calderón-Zygmund kernel as (1.1), $\omega^{q/p} \in A_1$, and $b \in Lip_{\beta,\omega}(0 < \beta < 1)$. Let $0 < \alpha + \beta < n, 1 < p < n/(\alpha+\beta)$ and $1/q = 1/p - (\alpha+\beta)/n$. If $T^1(f) = 0$ for any $f \in L^p(\omega)(1 < p < \infty)$, $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^p(\omega)$, $k = 1, \cdots, t$, then there exists a constant $C > 0$ such that,*

$$\|T^b(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})} \le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{L^p(\omega)}.$$

2 Some preliminaries

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B_r(x_0)$ denote the ball with the center x_0 and radius r, and $\lambda B = B_{\lambda r}(x_0)$ for any $\lambda > 0$. For a given