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On the Solution of Fractional Burgers' Equation and Its Optimal Control Problem

Svetlin G. Georgiev[1], Fatemeh Mohammadizadeh[2,*], Hojjat A. Tehrani[2] and M. H. Noori Skandari[2]

[1] *Faculty of Mathematical Science, Sorbonne University, Paris, France*
[2] *Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran*

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Abstract. The main aim of this study is to prove that a class of fractional Burgers' equations has a unique solution under some special conditions. Then it is demonstrated that an optimal control problem for this class of fractional Burgers' equations has at least one optimal solution.

Key Words: Burgers' equation, fractional optimal control, linear functional.

AMS Subject Classifications: 26A33, 34A08, 35R11

1 Introduction

Burgers' equation is a fundamental partial differential equation occurring in various areas of applied mathematics such as fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow [4,11].

Fractional Burgers' equation [9] describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in the other systems such as shallow-water waves and waves in bubbly liquids.

Some researchers have worked on solving this equation analytically and numerically. In [2], the fractional derivatives in the sense of the Jumaries modified Riemann-Liouville derivative of order alpha and the fractional complex transform are used to obtain the most general solution for the fractional Burgers' equation. Saad and Al-Sharif [14] applied the variational iteration method (VIM) to solve the time and space-time fractional

*Corresponding author. *Email addresses:* `f.mohamadi093@gmail.com` (F. Mohammadizadeh), `svetlingeorgiev1@gmail.com` (S. G. Georgiev), `hahsani@shahroodut.ac.ir` (H. A. Tehrani), `nooriskandari@shahroodut.ac.ir` (M. H. Noori Skandari)

Burgers' equation for various initial conditions. In [11], the fractional reduced differential transform method (FRDTM) is proposed to solve nonlinear fractional partial differential equations such as the space-time fractional Burgers' equations and the time-fractional Cahn-Allen equation. N. Khan et al. [5] used the generalized differential transform method (GDTM) and homotopy perturbation method (HPM) for solving time-fractional Burgers' and coupled Burgers' equations. N. Khan et al. [6] are used the Haar wavelet method to be obtained the numerical solutions of the time-fractional Schrödinger equations. The main advantages of the Haar wavelet method are its simple application and no requirement of residual or product operational matrix. In [7], some properties of Taylor's series with the heuristic optimization techniques are investigated some classes of time fractional convection-diffusion equations. Esen and Tasbozan [3] presented some numerical examples for the time fractional Burgers' equation with various boundary and initial conditions obtained by collocation method using cubic B-spline base functions.

This paper investigates an optimal control problem for a class of fractional Burgers' equations. The researcher proves that this problem has at least one optimal solution. Optimal control problems governed by time-dependent fractional partial differential equations play an important role in many practical problems such as production factor mobility, technological dissemination, pollution spreading and others. In technical applications, the fractional partial differential equations are not solved for their own sake but the solution should fulfill some requirements. In some sense, the application asks for an optimal solution of the problem.

This paper is organized as follows. In Section 2, we give some preliminaries which is needed to prove the main results in this paper. In Section 3, we introduce an optimal control problem for a class of fractional Burgers' equations and give two important results related to this problem. Finally, the conclusion is presented in Section 4.

2 Preliminary results

Definition 2.1 ([13])**.** Let $\mathbf{X}$ be a normed linear space. A linear functional $\mathbb{T}$ on $\mathbf{X}$ is said to be bounded if there is an $M \geq 0$ such that

$$|\mathbb{T}(f)| \leq M\|f\| \quad \text{for any } f \in \mathbf{X}. \tag{2.1}$$

The infimum of all such M is called the norm of $\mathbb{T}$ and it is denoted by $\|\mathbb{T}\|_\star$. The collection of bounded linear functionals on $\mathbf{X}$ is denoted by $\mathbf{X}^\star$ and is called the dual space of $\mathbf{X}$ which is a linear space.

Definition 2.2 ([13])**.** The linear operator $\mathbb{J} : \mathbf{X} \to (\mathbf{X}^\star)^\star$ defined by

$$\mathbb{J}(x)[\psi] = \psi(x) \quad \text{for all } x \in \mathbf{X}, \ \psi \in \mathbf{X}^\star, \tag{2.2}$$

is called the natural embedding of $\mathbf{X}$ into $(\mathbf{X}^\star)^\star$. Also, the space $\mathbf{X}$ is said to be reflexive when $\mathbb{J}(\mathbf{X}) = (\mathbf{X}^\star)^\star$. It is customary to denote $(\mathbf{X}^\star)^\star$ by $\mathbf{X}^{\star\star}$ and call $\mathbf{X}^{\star\star}$ the bidual of $\mathbf{X}$.