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A Note on Rough Parametric Marcinkiewicz Functions

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Abstract. In this note, we obtain sharp L^p estimates of parametric Marcinkiewicz integral operators. Our result resolves a long standing open problem. Also, we present a class of parametric Marcinkiewicz integral operators that are bounded provided that their kernels belong to the sole space $L^1\left(\mathbb{S}^{n-1}\right)$.

Key Words: Marcinkiewicz integrals, parametric Marcinkiewicz functions, rough kernels, Fourier transform, Marcinkiewicz interpolation theorem.

AMS Subject Classifications: 42B20, 42B15, 42B25

1 Introduction

Let $n \geq 2$ and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$ and

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{1.1}$$

In 1960, Hörmander (see [6]) introduced the following parametric Marcinkiewicz function μ_Ω^ρ of higher dimension by

$$\mu_\Omega^\rho f(x) = \Big(\int_{-\infty}^{\infty} \Big| 2^{-\rho t} \int_{|y|\leq 2^t} f(x-y)\,|y|^{-n+\rho}\,\Omega(y) dy \Big|^2 dt \Big)^{\frac{1}{2}}, \tag{1.2}$$

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where $\rho > 0$. When $\rho = 1$, the corresponding operator $\mu_\Omega = \mu_\Omega^1$ is the classical Marcinkiewicz integral operator introduced by Stein (see [7]). When $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$, $(0 < \alpha \le 1)$, Stein proved that μ_Ω is bounded on L^p for all $1 < p \le 2$. Subsequently, Benedek-Calderón-Panzone proved the L^p boundedness of μ_Ω for all $1 < p < \infty$ under the condition $\Omega \in C^1\left(\mathbb{S}^{n-1}\right)$ (see [4]). Since then, the L^p boundedness of μ_Ω has been investigated by several authors. For background information, we advise readers to consult [1–3,7], among others.

Concerning the problem whether there are some L^p results on μ_Ω^ρ similar to those on μ_Ω when Ω satisfies only some size conditions, Ding, Lu, and Yabuta (see [5]) studied the general operator

$$\mu^\rho_{\Omega,h} f(x) = \Big(\int_{-\infty}^{\infty} \left|2^{-\rho t}\int_{|y|\le 2^t} f(x-y)\,|y|^{-n+\rho}\,h(|y|)\Omega(y)dy\right|^2 dt\Big)^{\frac{1}{2}}, \tag{1.3}$$

where h is a radial function on $\mathbb{R}^n$ satisfying $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 \le q \le \infty$, where the class $l^\infty(L^q)(\mathbb{R}^+)$ is defined by

$$l^\infty(L^q)(\mathbb{R}^+) = \Big\{h : |h|_{l^\infty(L^q)(\mathbb{R}^+)} = \sup_{j\in\mathbb{Z}} \Big(\int_{2^{j-1}}^{2^j} |h(r)|^q \frac{dr}{r}\Big)^{\frac{1}{q}} < \infty\Big\}.$$

For $q = \infty$, we set $l^\infty(L^\infty)(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. It is clear that

$$l^\infty(L^\infty)(\mathbb{R}^+) \subset l^\infty(L^r)(\mathbb{R}^+) \subset l^\infty(L^q)(\mathbb{R}^+) \subset l^\infty(L^1)(\mathbb{R}^+),$$

$1 < q < r < \infty$. Ding, Lu, and Yabuta (see [5]) proved the following result:

Theorem 1.1 ([5]). *Suppose that $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1) *and $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$ for some $1 < q \le \infty$. If $Re(\rho) = \alpha > 0$, then $\left|\mu^\rho_{\Omega,h} f\right|_2 \le C\alpha^{-\frac{1}{2}}\,|f|_2$, where C is independent of ρ and f.*

In [1], Al-Salman and Al-Qassem considered the L^p boundedness of $\mu^\rho_{\Omega,h}$ for $p \ne 2$. which was left open in [5]. They proved the following result:

Theorem 1.2 ([1]). *Suppose that $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1). *If $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \le \infty$, and $\alpha = Re(\rho) > 0$, then $\left|\mu^\rho_{\Omega,h} f\right|_p \le C\alpha^{-1}\,|f|_p$ for all $1 < p < \infty$, where C is independent of ρ and f.*

In light of Theorem 1.1, it is clear that the dependence of the L^p bounds on α in Theorem 1.2 is not sharp. More precisely, we have the following long standing natural open problem:

Problem:

(a) Is the power $(-1/2)$ of α in Theorem 1.1 sharp?