Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 1 (2020), pp. 69-88

DOI: 10.4208/ata.OA-2018-0011

Existence and Uniqueness of Solution for a Class of Nonlinear Degenerate Elliptic Equations

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Received 21 November 2018; Accepted (in revised version) 29 September 2019

Abstract. In this work we are interested in the existence and uniqueness of solutions for the Navier problem associated to the degenerate nonlinear elliptic equations

$$\begin{aligned}&\Delta\Big[\omega_1(x)|\Delta u|^{p-2}\Delta u+v_1(x)|\Delta u|^{q-2}\Delta u\Big]\\&\quad-\operatorname{div}\Big[\omega_2(x)|\nabla u|^{r-2}\nabla u+v_2(x)|\nabla u|^{s-2}\nabla u\Big]\\=&f(x)-\operatorname{div}(G(x))\quad\text{in }\ \Omega,\end{aligned}$$

in the setting of the weighted Sobolev spaces.

Key Words: Degenerate nonlinear elliptic equation, Weighted Sobolev spaces.
AMS Subject Classifications: 35J60, 35J70

1 Introduction

In this work we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $X=W^{2,p}(\Omega,\omega_1)\cap W_0^{1,r}(\Omega,\omega_2)$ (see Definition 2.4 and Definition 2.5 for the Navier problem

$$(P)\qquad\begin{cases}Lu(x)=f(x)-\operatorname{div}(G(x)) & \text{in }\ \Omega,\\ u(x)=\Delta u(x)=0 & \text{on }\ \partial\Omega,\end{cases}$$

where L is the partial differential operator

$$\begin{aligned}Lu(x)=&\Delta\Big[\omega_1(x)|\Delta u|^{p-2}\Delta u+v_1(x)|\Delta u|^{q-2}\Delta u\Big]\\&-\operatorname{div}\Big[\omega_2(x)|\nabla u|^{r-2}\nabla u+v_2(x)|\nabla u|^{s-2}\nabla u\Big],\end{aligned}$$

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where Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, v_1 and v_2 are four weight functions, Δ is the Laplacian operator and $2 \le q, s < r < p < \infty$.

Let Ω be an open set in $\mathbb{R}^n$. We denote by $\mathcal{W}(\Omega)$ the set of all measurable, a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called weight functions. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus,

$$\mu_\omega(E) = \int_E \omega(x)dx$$

for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1–3, 5, 10] and [15]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [12]). These classes have found many useful applications in harmonic analysis (see [14]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [11]). There are, in fact, many interesting examples of weights (see [10] for p-admissible weights).

In the non-degenerate case (i.e., with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [9]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [3]), where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [13] and the references therein), and the degenerated p-Laplacian has been studied in [5]. The problem with degenerated p-Laplacian and p-Biharmonic operators in the case $\omega_1 = \omega_2 = v_1 = v_2$ and $p = q = r = s$

$$\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \text{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \text{div}(G(x)) & \text{in } \Omega, \\ u(x) = \Delta u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied by the author in [2].

The following theorem will be proved in Section 3.