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Inside the Light Boojums: a Journey to the Land of Boundary Defects

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Abstract. We consider minimizers of the energy

$$E_\varepsilon(u) =: \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2\right] dx + \frac{1}{2\varepsilon^s}\int_{\partial\Omega} W(u,g)ds, \ \ u:\Omega\to\mathbb{C}, \ \ 0<s<1,$$

in a two-dimensional domain Ω, with weak anchoring potential

$$W(u,g) =: \frac{1}{2}(|u|^2-1)^2 + (\langle u,g\rangle - \cos\alpha)^2, \quad 0<\alpha<\frac{\pi}{2}.$$

This functional was previously derived as a thin-film limit of the Landau-de Gennes energy, assuming weak anchoring on the boundary favoring a nematic director lying along a cone of fixed aperture, centered at the normal vector to the boundary.
In the regime where $s[\alpha^2+(\pi-\alpha)^2] < \pi^2/2$, any limiting map $u_*:\Omega\to\mathbb{S}^1$ has only boundary vortices, where its phase jumps by either 2α (light boojums) or $2(\pi-\alpha)$ (heavy boojums). Our main result is the fine-scale description of the light boojums.

Key Words: Nematics, thin-film limit, Ginzburg-Landau type energy, weak anchoring, boundary vortices, asymptotic profile.

AMS Subject Classifications: 35J20, 35B25, 35J61, 49K20

1 Introduction

In this paper, we consider minimizers of the following two-dimensional variational functional of Ginzburg-Landau type:

$$E_\varepsilon(u) =: \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2\right] dx + \frac{1}{2\varepsilon^s}\int_{\partial\Omega} W(u,g)\, ds. \tag{1.1}$$

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Here,

1. $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, supposed for simplicity simply connected.
2. $u : \Omega \to \mathbb{R}^2 \simeq \mathbb{C}$ belongs to the energy space $H^1(\Omega;\mathbb{C})$.
3. $W(u,g)$ is of the form
$$W(u,g) =: \frac{1}{2}(|u|^2-1)^2 + (\langle u,g\rangle - \cos\alpha)^2, \tag{1.2}$$
with $g : \partial\Omega \to \mathbb{S}^1$ is smooth, $\alpha \in (0,\pi/2)$ and $\langle\ ,\ \rangle$ the scalar product in $\mathbb{R}^2$.
4. $0 < s < 1$ is a parameter indicating the strength of the anchoring term $W(u,g)$.

This problem was derived in [2] as a thin-film limit of the Landau-de Gennes (Q-tensorial) model of nematic liquid crystals. Assuming the physical sample occupies a very thin cylinder over a planar domain $\Omega \subset \mathbb{R}^2$, and restricting to Q-tensors with a fixed eigenvalue in the vertical direction, Golovaty, Montero, & Sternberg [7] proved that the three-dimensional Landau-de Gennes energy Gamma converges to a two-dimensional Ginzburg-Landau functional. While the connection between nematics and the Ginzburg-Landau energy has been well established, the allure of (1.1) arises from the boundary condition imposed on minimizers, via energy penalization (or weak anchoring, in the parlance of liquid crystals). Instead of imposing a Dirichlet condition on the Q-tensor which forces the nematic director to align with the normal vector to the boundary, we may instead assume that it is energetically favorable for the director to lie along a cone of prescribed aperture to the normal. This may be modeled at the Landau-de Gennes level via a Rapini-Papoular surface energy [15], which in the thin-film limit appears as the boundary integral present in E_ε. The given function $g : \partial\Omega \to \mathbb{S}^1$, which may be chosen arbitrarily in the mathematical analysis of minimizers of E_ε, in the physical derivation of the model is given by the square $g = \nu^2$ of the complex representation of the outward unit normal vector $\nu = \nu_1 + \imath\nu_2$ to $\partial\Omega$. As $W(u,g) \geq 0$, with equality precisely when $u = g\,e^{\pm\imath\alpha}$, energy minimization favors u's which lie, on $\partial\Omega$, in the cone of aperture α around the vector g, and we expect to have $u \simeq ge^{\pm\imath\alpha}$ on $\partial\Omega$ most of the time.

The asymptotic analysis of the energy of minimizers of E_ε was undertaken in [2]. Using the bad discs construction as in [3, 14], adapted to problems with boundary penalization (see also [1, 12]), the authors derived a uniform upper bound on the energy of the minimizers u_ε in the complement of a finite number of small discs containing the defects. It follows from this preliminary analysis that any weak limit u_* of u_ε is smooth away from a finite defect set $\mathfrak{S}$, and satisfies $|u_*| = 1$ in $\overline{\Omega} \setminus \mathfrak{S}$ and $u_* = ge^{\pm\imath\alpha}$ on $\partial\Omega \setminus \mathfrak{S}$.

The novelty of this problem is that there are four classes of defects $\zeta \in \mathfrak{S}$ which might occur. As in the Dirichlet problem for Ginzburg-Landau, one may observe interior vortices, of integer degree. For boundary defects, there are three possibilities. First, u_* may jump from $u_* = ge^{+\imath\alpha}$ to $u_* = ge^{-\imath\alpha}$ (or from $ge^{-\imath\alpha}$ to $ge^{+\imath\alpha}$) across a defect $\zeta \in \partial\Omega$, by following the shortest path on $\mathbb{S}^1$, of length 2α. This type of defect is termed a light