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Recent Progress in the L_p Theory for Elliptic and Parabolic Equations with Discontinuous Coefficients

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Abstract. In this paper, we review some results over the last 10-15 years on elliptic and parabolic equations with discontinuous coefficients. We begin with an approach given by N. V. Krylov to parabolic equations in the whole space with VMO_x coefficients. We then discuss some subsequent development including elliptic and parabolic equations with coefficients which are allowed to be merely measurable in one or two space directions, weighted L_p estimates with Muckenhoupt (A_p) weights, non-local elliptic and parabolic equations, as well as fully nonlinear elliptic and parabolic equations.

Key Words: Elliptic and parabolic equations and systems, nonlocal equations, fully nonlinear equations, VMO and partially VMO coefficients, weighted estimates, Muckenhoupt weights.

AMS Subject Classifications: 35R05, 35B45, 35B65, 42B37, 35K20, 35J15, 35R11, 35K10, 35K45, 35J60, 35K55, 60J75

1 Introduction

The L_p-theory of elliptic and parabolic equations with discontinuous coefficients has been studied extensively for more than fifty years. On one hand, when the dimension $d = 2$, it is well known that the W_2^2 estimate holds for second-order uniformly elliptic operators with general bounded and measurable coefficients; see, for instance, Bernstein [5] and Talenti [108]. On the other hand, a celebrated counterexample in Talenti [107] and Maugeri et al. [97] indicates that when $d \geq 3$ in general there is no W_2^2 estimate for elliptic operators with bounded measurable coefficients even if they are discontinuous only at a single point. Another example due to Ural'tseva [110] shows the impossibility of the W_p^2 estimate when $d \geq 2$ and $p \neq 2$. See also Meyers [98] and Krylov [80]. Note that in Ural'tseva's example, the coefficients are continuous except at a single point when $d = 2$ or a line when $d = 3$. In [101], Nadirashvili showed that the weak uniqueness for martingale problems may fail if coefficients are merely measurable and $d \geq 3$. Recently,

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in [36] it was proved that when $p \neq 2$, there is no W^2_p estimate for elliptic operators with coefficients piecewise constant on each quadrant in $\mathbb{R}^2$. For divergence form equations, a similar estimate cannot be expected either due to an example by Piccinini and Spagnolo [103]. These examples imply that in general there does not exist a solvability theory for uniformly elliptic operators with bounded and measurable coefficients. Thus in the past many efforts have been made to treat particular types of discontinuous coefficients.

Among others, an important class of discontinuous coefficients is the class of vanishing mean oscillations (VMO). With VMO leading coefficients, the L_p-solvability theorems of second-order linear equations were established in early 1990s [7,14,15,21] for both divergence and nondivergence form equations. The main technical tool in these papers is the theory of singular integrals, in particular, the Calderón–Zygmund theorem and the Coifman–Rochberg–Weiss commutator theorem. However, this approach usually does not allow measurable coefficients because one needs smoothness of the corresponding fundamental solutions.

In 2005, Krylov [75] gave a unified kernel-free approach for both divergence and nondivergence linear elliptic and parabolic equations in the whole space, with coefficients which are in the class of VMO with respect to the space variables and are allowed to be merely measurable in the time variable. His proof relies on mean oscillation estimates, and uses the Hardy–Littlewood maximal function theorem and the Fefferman–Stein sharp function theorem. The results in [75] were shortly extended in [76] to the case of mixed-norm $L_p - L_q$ spaces, where the mixed norm is defined as $\|f\|_{q,p} = \|f\|_{L^t_q(L^x_p)}$. See also [55,74] for earlier results about the mixed-norm estimates for parabolic equations with coefficients independent of x, and the monograph [77]. Another approach was given earlier by Caffarelli and Peral [13], which is based on a level set argument together with a "crawling of ink spots" lemma originally due to Krylov and Safonov [70, 106] in the proof of the celebrated Krylov–Safonov C^α estimate for nondivergence form equations with measurable coefficients. See also Acerbi and Mingione [1] for a maximal function free argument applied to the parabolic p-Laplace equation, as well as [23, 57] for earlier work on the elliptic p-Laplace equation by using the sharp and maximal functions. With these approaches, VMO coefficients are treated in a straightforward manner by perturbation argument. Another advantage is that these approaches are rather flexible: they can be applied to both divergence and non-divergence or even non-local equations with coefficients which are very irregular in some of the independent variables. Rather than looking for as weak regularity assumptions as possible on coefficients, we emphasize that one of the main points of these work is to put forth new techniques which turn out to be more powerful than singular integrals in several occasions.

In this paper, we will first give an overview of Krylov's approach to parabolic equations in the whole space with VMO_x coefficients mentioned above in Section 2. Then we will review some subsequent important progress in this direction. In Section 3, we discuss divergence and nondivergence form elliptic and parabolic equations with partially and variably partially VMO coefficients. Section 4 is about a generalization of the