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# Some Estimates of Bilinear Hausdorff Operators on Stratified Groups

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**Abstract.** In this paper, we give four kinds of sharp estimates of two variants of bilinear Hausdorff operators on stratified groups, involving weighted Lebesgue spaces, classical Morrey spaces and central Morrey spaces. Meanwhile, some necessary and sufficient conditions of boundness are obtained.

**Key Words**: Stratified group, Hausdorff operator, bilinear, sharp estimate, weighted Lebesgue space, Morrey space.

**AMS Subject Classifications**: 42B35, 43A80, 47H60, 47A30

## 1 Introduction

The investigation on Hausdorff operators is a branch of the classical analysis and it is closely related to Fourier analysis and complex analysis. The Hausdorff summability methods was introduced [18, 19] long time ago to solve some classical problems. Readers can refer to [31] for the Hausdorff summability and its application and [17] for the Hausdorff means and its application on the summation of series. For more details of the background and the historical development of the operators, one can see [24]. During modern time, studies focusing on the Hausdorff operators have never stopped and the operator became its modern versions which were initiated by Siskakis in the work on complex analysis [30] and Georgakis [13], Liflyand and Móricz [22, 23] in their work on the Fourier transform setting. Readers can also refer to [3, 21, 27], for example, for more results about Hausdorff operators right after them.

The classical 1-dimension Hausdorff operator is defined by (see [22])

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\Big(\frac{x}{t}\Big) dt, \quad x \in \mathbb{R},$$

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where $\Phi$ is a locally integrable function in $(0,\infty)$. Using variables substitution, the above formula is rewritten as the following equality

$$h_\Phi(f)(x)=\int_0^\infty \frac{\Phi(x/t)}{t}f(t)dt.$$

We can take appropriate function $\Phi$ to obtain many classical operators in analysis as its special cases. These operators include Hardy operators, Cesàro operators, Hilbert operators, Hardy-Littlewood-Pólya operaotrs and others. We shall list [10,26,30] here for instance. Three extensions of the 1-dimensional Hausdorff operator in $\mathbb{R}^n$ were introduced and studied in [4,21] respectively. One of them is the operator [4]

$$H_\Phi f(x)=\int_{\mathbb{R}^n}\frac{\Phi(|y|^{-1}x)}{|y|^n}f(y)dy,$$

where $\Phi$ is a radial function defined on $\mathbb{R}^+$. With $\Phi_1(t)=t^{-n}\chi_{(1,\infty)}(t)$, $\Phi_2=t^{-n}\chi_{(0,1]}(t)$, $H_\Phi$ becomes the high dimensional Hardy operator $\mathcal{H}$ and its adjoint operator which is defined as

$$\mathcal{H}^*(f)(x)=\int_{|y|\ge|x|}f(y)dy.$$

We have noticed that $H_\Phi$ can map functions on $\mathbb{R}^{nm}$ to functions on $\mathbb{R}^n$ in some ways. In [5], Chen, Fan and Zhang introduced such a extension of $H_\Phi$, i.e.,

$$T_\Phi(F)(x)=\int_{\mathbb{R}^{nm}}\frac{\Phi(|u|^{-1}x)}{|u|^{nm}}F(u)du,$$

where $x\in\mathbb{R}^n$. Then taking $m=2$, $u=(u_1,u_2)$, $u_i\in\mathbb{R}^n$, $i=1,2$, and $F(u_1,u_2)=f_1(u_1)f_2(u_2)$, we have

$$T_\Phi(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\frac{\Phi(|u|^{-1}x)}{|u|^{2n}}f_1(u_1)f_2(u_2)du.$$

The above operator is called bilinear Hausdorff operator and can reduce to bilinear Hardy operator (see [10]) when we take

$$\Phi(t)=\frac{1}{\Omega^{2n}}t^{-2n}\chi_{(1,\infty)}(t).$$

Meanwhile, they [5] defined another version of the multilinear Hausdorff operator and the following is the bilinear case

$$S_\Psi(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\frac{\Psi(|u_1|^{-1}x,|u_2|^{-1}x)}{|u_1|^n\,|u_2|^n}f_1(u_1)f_2(u_2)du,$$

where $\Psi(s_1,s_2)$ is a locally integrable function on $\mathbb{R}^+\times\mathbb{R}^+$. Let

$$\Psi(s_1,s_2)=\Psi_1(s_1,s_2)=(s_1s_2)^{-n}\chi_{s_1^{-2}+s_2^{-2}<1}(s_1,s_2),$$