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On Sharpening of a Theorem of Ankeny and Rivlin

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Abstract. Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n,

$$M(p,R) =: \max_{|z|=R\geq 0} |p(z)| \quad \text{and} \quad M(p,1) =: \|p\|.$$

Then according to a well-known result of Ankeny and Rivlin [1], we have for $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\|.$$

This inequality has been sharpened by Govil [4], who proved that for $R\geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2-4|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\,\|p\|}{\|p\|+2|a_n|} - \ln\left(1+\frac{(R-1)\,\|p\|}{\|p\|+2|a_n|}\right)\right\}.$$

In this paper, we sharpen the above inequality of Govil [4], which in turn sharpens the inequality of Ankeny and Rivlin [1].

Key Words: Inequalities, polynomials, zeros.

AMS Subject Classifications: 15A18, 30C10, 30C15, 30A10

1 Introduction

Let $p(z)=\sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n, and for $R\geq 0$, let

$$M(p,R) =: \max_{|z|=R} |p(z)|.$$

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We will denote $M(p,1)$ by $\|p\|$. Then, by the maximum modulus principle $M(p,R)$ is a strictly increasing function of R and is defined for $0 \leq R < \infty$. Also, it is a simple deduction from the maximum modulus principle (see [11, pp. 158, Problem 269]) that for $R \geq 1$,

$$M(p,R) \leq R^n \|p\|. \tag{1.1}$$

The result is best possible and equality holds if and only if $p(z) = \lambda z^n$, λ being a complex number.

For polynomials of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [1] sharpened inequality (1.1), by proving following result.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \Big(\frac{R^n+1}{2}\Big) \|p\|, \quad R \geq 1. \tag{1.2}$$

The above inequality is sharp and equality holds for polynomials having all their zeros on the unit circle.

Several papers and research monographs have been written on this subject (see, for example Frappier, Rahman and Ruscheweyh [2], Gardner, Govil and Weems [3], Govil [5], Govil, Qazi and Rahman [6], Milovanović, Mitrinović and Rassias [8], Nwaeze [10], Rahman and Schmeisser [13,14], Sharma and Singh [15], and Zireh [16]).

A refinement of the above inequality (1.2) was given by Govil [4], who proved

Theorem 1.2. *If $p(z)$ is a polynomial of degree n, and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right)\Bigg\{ & \frac{(R-1)\|p\|}{\|p\|+2|a_n|} \\ & - \ln\left(1 + \frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right)\Bigg\}. \end{aligned} \tag{1.3}$$

The result is best possible and the equality holds for $p(z) = (\lambda + \mu z^n)$, where λ and μ are complex numbers with $|\lambda| = |\mu|$.

2 Main results

In this paper, we prove the following result which sharpens the above Theorem 1.2 due to Govil [4], and so in turn Theorem 1.1 due to Ankeny and Rivlin [1].

Theorem 2.1. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$ and any N, $1 \leq N \leq n$,*

$$M(p,R) \leq \frac{(R^n+1)}{2}\|p\| - \frac{n}{2}\|p\|\left(1 - \frac{2|a_n|}{\|p\|}\right) h(N), \tag{2.1}$$