

Revisit of Semi-Implicit Schemes for Phase-Field Equations

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. It is a very common practice to use semi-implicit schemes in various computations, which treat selected linear terms implicitly and the nonlinear terms explicitly. For phase-field equations, the principal elliptic operator is treated implicitly to reduce the associated stability constraints while the nonlinear terms are still treated explicitly to avoid the expensive process of solving nonlinear equations at each time step. However, very few recent numerical analysis is relevant to semi-implicit schemes, while "stabilized" schemes have become very popular. In this work, we will consider semi-implicit schemes for the Allen-Cahn equation with *general potential* function. It will be demonstrated that the maximum principle is valid and the energy stability also holds for the numerical solutions. This paper extends the result of Tang & Yang (J. Comput. Math., 34(5) (2016), pp. 471–481), which studies the semi-implicit scheme for the Allen-Cahn equation with *polynomial potentials*.

Key Words: Semi-implicit, phased-field equation, energy dissipation, maximum principle.

AMS Subject Classifications: 65M06, 65M12

1 Introduction

There has been tremendous interests in developing energy-dissipation numerical methods for phase-field models starting from earlier numerical works [1, 6, 12]. To make sure a numerical scheme satisfies the nonlinear energy stability, there are basically three class of approaches. For ease of exposition, we consider the simplest phase-field model, i.e.,

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the Allen-Cahn equation with initial condition:

$$\frac{\partial \phi}{\partial t} = \varepsilon^2 \Delta \phi - f(\phi) \quad \text{in } \Omega \times (0, T], \quad (1.1a)$$

$$\phi(x, 0) = \phi_0(x) \quad \text{in } \Omega, \quad (1.1b)$$

where $\varepsilon > 0$ is the interface width parameter, $f(\phi) = F'(\phi)$, where F is a smooth function.

The corresponding energy is defined as

$$E(\phi) =: \int_{\Omega} \left(\frac{\varepsilon^2}{2} |\nabla \phi|^2 + F(\phi) \right) dx. \quad (1.2)$$

Energy stability means that

$$E(\phi(\cdot, t)) \leq E(\phi(\cdot, s)), \quad \forall t > s. \quad (1.3)$$

The first class of energy stable scheme is the Eyre's convex splitting method [6], which yields a *nonlinear* semi-implicit scheme:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \varepsilon^2 \Delta \phi^{n+1} - T_1(\phi^{n+1}) - T_2(\phi^n), \quad (1.4)$$

where T_1 and T_2 are some convex functionals satisfying $T_1 + T_2 = f$. This is also referred as partially implicit scheme for phase-field modeling by [13].

The second class of energy stable scheme is to add some extra terms so that the resulting scheme satisfies energy non-increasing property; these schemes are called "stabilized" approach. There have been quite large size of papers in this direction in the past 15 years, see the review articles [9, 10] and references therein.

The third class is the direct fully implicit scheme, see, e.g., [5, 8, 13]. In particular, it is demonstrated in [13] that a first-order fully implicit scheme for the Allen-Cahn model can be devised so that the maximum principle is valid on the discrete level and, furthermore, the linearized discretized system can be effectively preconditioned using discrete Poisson operators.

It is noted by [6] that an unconditionally energy-stable scheme, such that backward Euler scheme, is not necessarily better than a conditionally energy stable scheme when the time step size is not small enough. In other words, if larger time steps are needed then the first and second classes schemes are useful. However, it is argued in [13] that most implicit schemes are energy-stable if the time-step size is sufficiently small. Moreover, it is noted that a convex splitting scheme or a "stabilized" approach can be equivalent to some fully implicit scheme with a different time scaling and thus it may lack numerical accuracy.

It is obvious that the partially implicit scheme and the fully implicit scheme all require some iteration techniques, and are less effective than the explicit scheme or semi-implicit