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Lyapunov Center Theorem of Infinite Dimensional Hamiltonian Systems

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. In this paper we reformulate a Lyapunov center theorem of infinite dimensional Hamiltonian systems arising from PDEs. The proof is based on a modified KAM iteration for periodic case.

Key Words: Hamiltonian systems, KAM iteration, small divisors, lower dimensional tori.

AMS Subject Classifications: 37K55, 35B10, 35J10, 35Q40

1 Introduction

The KAM theory of infinite dimensional Hamiltonian systems arising from PDEs has been studied widely. Earlier works in this area are due to Wayne, Craig, Bourgain, Kuksin, Pöschel, Eliasson and etc. [4, 6–8, 11, 13, 15]. More recently, the infinite dimensional KAM theory has been developed and there are many infinite dimensional KAM-type theorems. For some related results, see [1–3, 5, 9, 10, 12, 14, 18] and the references therein. We also refer to [17] for a survey on both finite and infinite dimensional KAM theory.

However, most of previous works mainly focus on quasi-periodic case, since periodic case can be regarded as a special quasi-periodic case. But Craig and Wayne constructed periodic solutions of the nonlinear wave equations and nonlinear Schrödinger equations with periodic boundary value by the Nash-Moser methods in weaker small divisor conditions, see [6, 7]. Their results give an infinite dimensional version of the Lyaponuv center theorem.

In this paper, we want to extend the finite dimensional Lyapunov center theorem to infinite dimensional Hamiltonian systems. The result can be regarded as a complement of infinite dimensional KAM theorems. Our results can also be applied to some partial differential equations, but here we do not pursue this problem.

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2 Main results

Consider a nearly integrable infinite dimensional Hamiltonian $H = N + P$. The normal form

$$N = \omega I + \frac{1}{2}\sum_{j=1}^{\infty}\Omega_j(\omega)(u_j^2 + v_j^2), \quad (\theta, I, u, v) \in \Gamma^{a,p} = \mathbb{T} \times \mathbb{R} \times \ell^{a,p} \times \ell^{a,p},$$

where $\mathbb{T}$ is the usual 1-torus, $\mathbb{R}$ is the 1-dimensional real space and the Hilbert space

$$\ell^{a,p} = \{w = (w_1, w_2, \cdots) | \|w\|_{a,p} < +\infty\}$$

with

$$\|w\|_{a,p}^2 = \sum_{j=1}^{\infty} |w_j|^2 j^{2p} e^{2aj}, \quad a \geq 0, \quad p \geq 0.$$

The frequency $\omega \in O = (\beta_1, \beta_2)$ is regarded as parameter and the normal frequencies $\Omega_1, \Omega_2, \cdots$ are Lipschitz-continuous in the parameter ω.

The small perturbation $P = P(\theta, I, u, v)$ is analytic in (θ, I, u, v) and Lipschitz-continuous in ω, here and below, the dependence of P in the parameter ω is usually implied and not written explicitly only for simplicity if there is no any confusion.

The Hamiltonian system of H is

$$\dot{\theta} = H_I = \omega + P_I, \qquad \dot{I} = -H_\theta = -P_\theta, \tag{2.1a}$$
$$\dot{u} = H_v = \Omega v + P_v, \qquad \dot{v} = -H_u = -\Omega u - P_u, \tag{2.1b}$$

where $\Omega = (\Omega_1, \Omega_2, \cdots)$ and $\Omega u = (\Omega_1 u_1, \Omega_2 u_2, \cdots)$.

If $P = 0$, the system (2.1) is integrable and admits a family of invariant tori

$$\mathcal{T}_\omega = \mathbb{T} \times \{0\} \times \{0\} \times \{0\} \subset \Gamma^{a,p}, \quad \forall \omega \in O,$$

on which there are periodic trajectories

$$\theta = \omega t + \theta_0, \quad \forall \theta_0 \in \mathbb{T}, \quad I = 0, \quad u = 0, \quad v = 0.$$

If $P \neq 0$, the system (2.1) is not integrable in general. If the space $\ell^{a,p}$ is finite dimensional, the Lyaponuv center theorem says, if $k\omega \pm \Omega_j \neq 0, \forall k \in \mathbb{Z}, \forall j \geq 1$, and the perturbation P is sufficiently small, then the Hamiltonian system (2.1) has many 1-dimensional invariant tori with the frequency ω, which are embeddings of $\mathcal{T}_\omega$ in $\Gamma^{a,p}$.

In this paper we will prove a similar result in infinite dimensional case. To state our main results, we first introduce some notations.

Let $f(\theta)$ be a 2π-periodic function and its Fourier series expansion is

$$f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{\mathrm{i}k\theta}.$$