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A Survey on Some Anisotropic Hardy-Type Function Spaces

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. Let A be a general expansive matrix on $\mathbb{R}^n$. The aims of this article are twofold. The first one is to give a survey on the recent developments of anisotropic Hardy-type function spaces on $\mathbb{R}^n$, including anisotropic Hardy–Lorentz spaces, anisotropic variable Hardy spaces and anisotropic variable Hardy–Lorentz spaces as well as anisotropic Musielak–Orlicz Hardy spaces. The second one is to correct some errors and seal some gaps existing in the known articles. Some unsolved problems are also presented.

Key Words: Expansive matrix, (variable) Hardy space, (variable) Hardy–Lorentz space, Musielak–Orlicz Hardy space.

AMS Subject Classifications: 42B35, 42B30, 42B25, 46B70

1 Introduction

In order to meet the requirements arising in the development of harmonic analysis and partial differential equations, there has been more and more research in extending classical function spaces from Euclidean spaces to some more general underlying spaces; see, for instance, [8, 34, 44, 47, 49, 58, 85, 90, 116]. In 2003, to give a unified framework of the real-variable theory of both the isotropic Hardy space and the parabolic Hardy space of Calderón and Torchinsky [19], for the first time, Bownik [12] introduced the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$ with $p \in (0, \infty)$, where A is a general expansive matrix on $\mathbb{R}^n$ (see [12, p. 5, Definition 2.1]). In [12], Bownik also established the characterizations

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of $H^p_A(\mathbb{R}^n)$, respectively, in terms of atoms, maximal functions and tight frame multiwavelets (see [12, p. 94, Definition 4.2]), and proved as well that the dual space of $H^p_A(\mathbb{R}^n)$ with $p \in (0,1]$ is the anisotropic Campanato space; as applications, Bownik [12] also obtained the boundedness of anisotropic Calderón–Zygmund operators from $H^p_A(\mathbb{R}^n)$ to itself (or to the Lebesgue space $L^p(\mathbb{R}^n)$). Later on, Bownik et al. [16] further extended the anisotropic Hardy space to the weighted setting. Very recently, Wang [110] considered a multiplier theorem on anisotropic Hardy spaces $H^p_A(\mathbb{R}^n)$. Nowadays, the anisotropic setting has proved useful not only in developing function spaces, but also in many other branches such as the wavelet theory (see, for instance, [5, 12, 25]) and partial differential equations (see, for instance, [18, 53]).

Let us briefly recall some history of the study of anisotropic function spaces. It has been developed parallel to the theory for isotropic spaces; we refer the reader in particular to the monographs [9, 88] (and the articles mentioned there), and to the survey [10]. For any $p \in (1,\infty)$ and $\{s_i\}_{i=1}^n \subset \mathbb{N}$, the (classical) anisotropic Sobolev space on $\mathbb{R}^n$ contains all $f \in L^p(\mathbb{R}^n)$ such that

$$\frac{\partial^{s_i} f}{\partial x_i^{s_i}} \in L^p(\mathbb{R}^n) \quad \text{for any } \ i \in \{1,\cdots,n\}.$$

It is obvious that, unlike in case of the isotropic Sobolev space (namely, the case when $s_1 = \cdots = s_n$), the smoothness properties of an element depend on the chosen direction in $\mathbb{R}^n$. The number s, defined by setting $\frac{1}{s} := \frac{1}{n}(\frac{1}{s_1} + \cdots + \frac{1}{s_n})$, is usually called the mean smoothness, and $a = (a_1,\cdots,a_n)$, given by $a_i := \frac{s}{s_i}$, $i \in \{1,\cdots,n\}$, characterizes the anisotropy. Similarly to the isotropic situation, more general scales of anisotropic Bessel potential spaces (fractional Sobolev spaces), anisotropic Besov spaces and anisotropic Triebel–Lizorkin spaces were studied. It is well known that the isotropic theory has a more or less complete counterpart of the fundamentals (definitions, description via differences and derivatives, elementary properties, embeddings for different metrics, interpolation) in the context of anisotropic spaces. A survey on the basic results for the (anisotropic) spaces of Besov or Triebel–Lizorkin type was given in [94, Subsections 4.2.1 through 4.2.4] (with preceding results in [83, 92, 93, 103–106]) and [57, Sections 2.1 and 2.2]. More recently, several authors were concerned with the problem of obtaining useful decompositions of anisotropic function spaces of Besov and Triebel–Lizorkin type. A construction of unconditional bases using Meyer wavelets was obtained in [7, 8]; see also [44, 45, 49]; a different approach, involving the φ-transform of Frazier and Jawerth (see [40, 41]) was followed in [28–30]; see also [95]. More recent contributions can be found in [13–15] and [55, 61, 62]. Based on the approach used in [107, 108], further representations were obtained by local means, atomic and sub-atomic decompositions, which can be found in [34, 47]; see also [24, 35, 36, 101, 112, 113] for applications. Finally, let us refer the reader to [109, Chapter 5] where Triebel gave a very nice and detailed summary of the history, recent developments and the state-of-the-art (in 2006), which we also recommend for further references. Moreover, Barrios et al. [4] further characterized the anisotropic Besov spaces in terms of Peetre maximal functions and approximations; Li et