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Lipschitz Invariance of Critical Exponents on Besov Spaces

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

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Abstract. In this paper we prove that the critical exponents of Besov spaces on a compact set possessing an Ahlfors regular measure is an invariant under Lipschitz transforms. Under mild conditions, the critical exponent of Besov spaces of certain self-similar sets coincides with the walk dimension, which plays an important role in the analysis on fractals. As an application, we show examples having different critical exponents are not Lipschitz equivalent.

Key Words: Lipschitz invariant, Besov space, critical exponents, walk dimension, heat kernel.

AMS Subject Classifications: 35K08, 28A80, 35J08, 46E35, 47D07

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1 Introduction

Let (X,d_1) and (Y,d_2) be two metric spaces. We say that $T:(X,d_1)\to(Y,d_2)$ is a bi-Lipschitz transform, if T is a bijection and furthermore, there exists a constant $C>0$ such that for any $x,y\in X$,

$$C^{-1}d_1(x,y)\le d_2(Tx,Ty)\le Cd_1(x,y). \tag{1.1}$$

The study of Lipschitz equivalence of self-similar sets are initiated by Falconer and Marsh [5] and David and Semmes [4]. Rao, Ruan and Xi [18] (2006) answered a question posed by David and Semmes [4], by showing that the self-similar sets in Fig. 1 are Lipschitz equivalent. After that, there are many works devoted to this topic, for example, Xi and Xiong [24,25], Luo and Lau [15], Ruan et al. [22], and Rao and Zhang [19]. However, the studies mentioned above are all on self-similar sets with simple topological structure, that is, the fractals under consideration are totally disconnected.

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Recently, there are some studies on a class of self-similar sets which are not totally disconnected. A non-empty compact set satisfying the set equation $F = \bigcup_{d\in\mathcal{D}} \frac{F+d}{n}$ is called a fractal square if $n \geq 2$ and $\mathcal{D} \subset \{0,1,\cdots,n-1\}^2$. Ruan and Wang [21] studied fractal squares of ratio $1/3$ and with 7 or 8 branches, which are all connected fractals. Their method is to find various connectivity properties, which depend on very careful observations. Rao and Zhu [20] and Zhu and Yang [26] construct bi-Lipschitz mappings between the fractals illustrated in Fig. 3.

Figure 1: The Cantor sets E and F are Lipschitz equivalence [18].

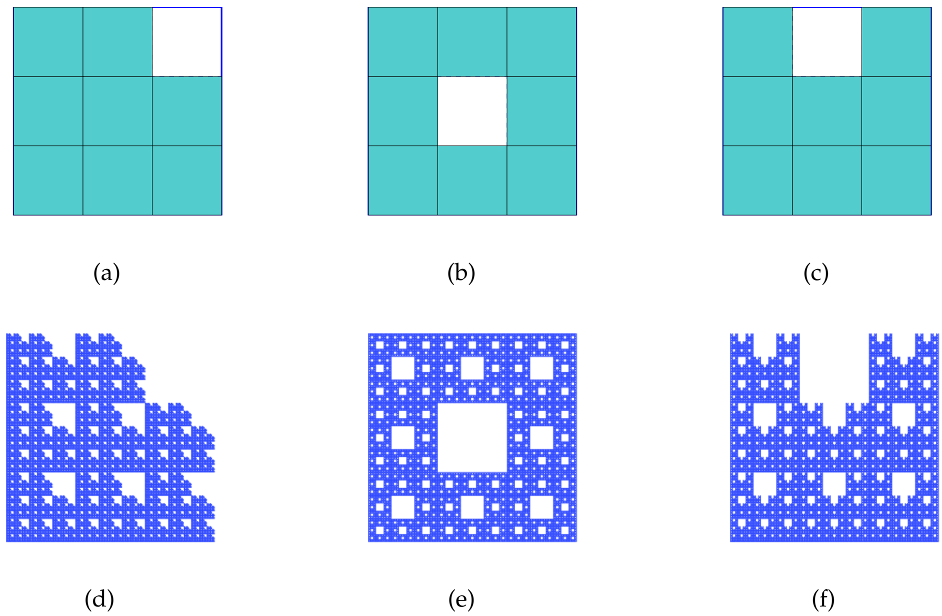

Figure 2: [21] shows that the above three fractal squares are not Lipschitz equivalent.

The study of the Lipschitz equivalence of connected self-similar sets is a very hard problem. Let us consider the fractals in Fig. 4. By a result of Whyburn [23], they are homeomorphic, and [21] conjectures these two fractal squares are not Lipschitz equivalent (see also [17]). To show two sets are not Lipschitz equivalent, the main method is to construct a certain Lipschitz invariant to distinct them.

In this paper, we use the Besov spaces to construct Lipschitz invariants for fractals