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Hausdorff Dimension of a Class of Weierstrass Functions

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. It was proved by Shen that the graph of the classical Weierstrass function $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$ has Hausdorff dimension $2 + \log\lambda/\log b$, for every integer $b \geq 2$ and every $\lambda \in (1/b, 1)$ [Hausdorff dimension of the graph of the classical Weierstrass functions, Math. Z., 289 (2018), 223–266]. In this paper, we prove that the dimension formula holds for every integer $b \geq 3$ and every $\lambda \in (1/b, 1)$ if we replace the function cos by sin in the definition of Weierstrass function. A class of more general functions are also discussed.

Key Words: Hausdorff dimension, Weierstrass function, SRB measure.

AMS Subject Classifications: 28A80

1 Introduction

Weierstrass functions are classical fractal functions. The non-differentiability of these functions were studied by Weierstrass and Hardy [2]. Recently, Shen [7] proved that the graph of the classical Weierstrass function $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$ has Hausdorff dimension $2 + \log\lambda/\log b$, for every integer $b \geq 2$ and every $\lambda \in (1/b, 1)$, which solved a long-standing conjecture. Some relevant results can be found in [1, 3–5, 8]. Naturally, we want to study the Hausdorff dimension of the graph of Weierstrass functions with the following form:

$$W_{\lambda,b,\theta}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x + \theta), \quad x \in \mathbb{R},$$

where $b \geq 2$ is an integer, $\lambda \in (1/b, 1)$ and $\theta \in \mathbb{R}$.

Denote $D_{\lambda,b} = 2 + \log\lambda/\log b$. Denote by $\dim_{\mathrm{H}} \Gamma W_{\lambda,b,\theta}$ the Hausdorff dimension of the graph of $W_{\lambda,b,\theta}$. Our main result is:

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Theorem 1.1. *If $\theta = -\pi/2$, then $\dim_{\rm H} \Gamma W_{\lambda,b,\theta} = D_{\lambda,b}$ for every integer $b \geq 3$ and every $\lambda \in (1/b, 1)$. If the integer $b \geq 7$, then the dimension formula holds for every $\lambda \in (1/b, 1)$ and every $\theta \in \mathbb{R}$.*

The paper is organized as follows. In next section, we present necessary notations and properties introduced by Shen [7] and Tsujii [8]. In Sections 3 and 4, we prove the main result.

2 Preliminaries

In this section, we present necessary notations and properties introduced in [7,8]. Denote $\gamma = 1/(\lambda b)$, $\phi_\theta(x) = \cos(2\pi x + \theta)$, and $\psi_\theta(x) = \phi'_\theta(x)$. Let $\mathcal{A} = \{0, 1, \cdots, b-1\}$. Given $x \in \mathbb{R}$ and $\mathbf{u} = \{u_n\}_{n=1}^\infty \in \mathcal{A}^{\mathbb{Z}^+}$, we define

$$S_\theta(x, \mathbf{u}) = \sum_{n=1}^{\infty} \gamma^{n-1} \psi_\theta(x(\mathbf{u}|_n)),$$

where $\mathbf{u}|_n = (u_1, \cdots, u_n)$ and

$$x(\mathbf{u}|_n) = \frac{x}{b^n} + \frac{u_1}{b^n} + \frac{u_2}{b^{n-1}} + \cdots + \frac{u_n}{b}.$$

For simplicity, we will use $S(x, \mathbf{u})$ to denote $S_\theta(x, \mathbf{u})$ if no confusion occurs.

Given $\varepsilon, \delta > 0$. Two words $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{Z}^+}$ are called *(ε, δ)-tangent* at a point $x_0 \in \mathbb{R}$ if

$$|S(x_0, \mathbf{i}) - S(x_0, \mathbf{j})| \leq \varepsilon \quad \text{and} \quad |S'(x_0, \mathbf{i}) - S'(x_0, \mathbf{j})| \leq \delta.$$

Let $E(q, x_0; \varepsilon, \delta)$ denote the set of pairs $(\mathbf{k}, \mathbf{l}) \in \mathcal{A}^q \times \mathcal{A}^q$ for which there exist $\mathbf{u}, \mathbf{v} \in \mathcal{A}^{\mathbb{Z}^+}$ such that $\mathbf{ku}$ and $\mathbf{lv}$ are (ε, δ)-tangent at x_0. Let

$$\begin{aligned} e(q, x_0; \varepsilon, \delta) &= \max_{\mathbf{k} \in \mathcal{A}^{\mathbb{Z}^+}} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0; \varepsilon, \delta)\}, \\ E(q, x_0) &= \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, x_0; \varepsilon, \delta), \\ e(q, x_0) &= \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0)\}. \end{aligned}$$

For $J \subset \mathbb{R}$, define

$$\begin{aligned} E(q, J; \varepsilon, \delta) &= \bigcup_{x_0 \in J} E(q, x_0; \varepsilon, \delta), \\ E(q, J) &= \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, J; \varepsilon, \delta), \\ e(q, J) &= \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, J)\}. \end{aligned}$$