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Local Well-Posedness for the Compressible Nematic Liquid Crystals Flow with Vacuum

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. In this paper we prove the local well-posedness of strong solutions to the compressible nematic liquid crystals flow with vacuum in a bounded domain $\Omega \subset \mathbb{R}^3$.

Key Words: Liquid crystals, vacuum, Local well-posedness, strong solution.

AMS Subject Classifications: 76D03, 35Q30, 35Q35

1 Introduction

In this paper we consider the following simplified version of the Ericksen-Leslie system modeling the flow of compressible nematic liquid crystals (see [2,3]):

$$\partial_t\rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1a}$$
$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = -\nabla d \cdot \Delta d, \tag{1.1b}$$
$$\partial_t d + u\cdot\nabla d - \Delta d = d|\nabla d|^2 \quad \text{in } \Omega\times(0,T), \tag{1.1c}$$

with boundary and initial conditions:

$$u=0, \quad \frac{\partial d}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{1.2a}$$
$$(\rho,u,d)(\cdot,0) = (\rho_0,u_0,d_0)(\cdot) \qquad \text{in } \Omega\subset\mathbb{R}^3, \tag{1.2b}$$

where $\rho \geq 0$ is the density of the fluid, $u\in\mathbb{R}^3$ represents velocity field of the fluid, $d\in\mathbb{S}^2$ represents the macroscopic average of the nematic liquid crystals orientation field. The

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parameters μ and λ are shear viscosity and the bulk viscosity coefficients of the fluid, respectively, satisfying the physical conditions:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

We assume that the pressure p satisfies the γ-law, i.e., $p =: a\rho^\gamma$ with constants $a > 0$ and $\gamma > 1$. The domain $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, and n is the unit outward normal vector to $\partial\Omega$.

Below let us review some results to the system (1.1a)-(1.1c) briefly. Ding et al. [2] first introduced the system (1.1a)-(1.1c) and studied the low Mach number limit of it, see [9–11] on the recent progress on this topic. Huang, Wang, and Wen [3] (see also [1,5]) showed the local well-posedness of strong solutions with vacuum under the following compatibility condition:

$$-\mu\Delta u_0 - (\lambda+\mu)\nabla \mathrm{div}\, u_0 + \nabla(a\rho_0^\gamma) + \nabla d_0 \cdot \Delta d_0 = \sqrt{\rho_0} g \tag{1.3}$$

for some $g \in L^2(\Omega)$. Jiang, Jiang, and Wang [4] (see also [6]) proved the global existence of weak solutions in $\mathbb{R}^2$. Lin, Lai and Wang [7] established the existence of global weak solutions with finite energy and density satisfying the renormalized continuity equation, provided the initial orientation director field lies in the hemisphere $\mathbb{S}^2_+$.

The purpose of this paper is to establish the local well-posedness of strong solutions of the compressible nematic liquid crystal model (1.1a)-(1.1c) without the compatibility condition (1.3).

We will prove

Theorem 1.1. *Let* $0 \leq \rho_0 \in W^{1,q}$, $(3 < q < 6)$, $u_0 \in H_0^1$, $d_0 \in H^2$ *with* $|d_0| = 1$. *Then the problem* (1.1a)-(1.2b) *has a unique local strong solution* (ρ, u, d) *satisfying*

$$\begin{cases} \rho \in C([0,T];L^2) \cap L^\infty(0,T;W^{1,q}), & \partial_t\rho \in L^\infty(0,T;L^2), \\ \rho u \in C([0,T];L^2),\ u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2), & \sqrt{\rho}\partial_t u \in L^2(0,T;L^2), \\ \sqrt{t} u \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,q}), & \sqrt{t}\partial_t u \in L^2(0,T;H_0^1), \\ d \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3),\ \partial_t d \in L^2(0,T;H^1), & \sqrt{t}\partial_t d \in L^\infty(0,T;H^1), \end{cases} \tag{1.4}$$

for some $0 < T \leq \infty$.

We will prove Theorem 1.1 in the following way: For $\delta > 0$, we choose $0 < \delta \leq \rho_0^\delta \in H^2$ and $u_0^\delta \in H_0^1 \cap H^2$ satisfying

$$\rho_0^\delta \to \rho_0 \quad \text{in } W^{1,q} \quad \text{and} \quad u_0^\delta \to u_0 \quad \text{in } H_0^1 \quad \text{as } \delta \to 0. \tag{1.5}$$

Then it is easy to verify that the problem (1.1a)-(1.2b) has a unique local strong solution $(\rho^\delta, u^\delta, d^\delta)$ in $[0, T_\delta)$.