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The Non-Degeneracy of Harmonic Structures on Planar Sierpinski Gaskets

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. We present a direct and short proof of the non-degeneracy of the harmonic structures on the level-n Sierpinski gaskets for any $n \geq 2$, which was conjectured by Hino in [1, 2] and confirmed to be true by Tsougkas [8] very recently using Tutte's spring theorem.

Key Words: Fractal analysis, harmonic functions, fractal Laplacians, harmonic structures, Sierpinski gaskets.

AMS Subject Classifications: 28A80

1 Introduction

The theory of analysis on fractals, analogous to that on manifolds, has been being well developed. The pioneering work, developed by Kigami [3, 4], is the analytic construction of the Laplacians, for a class of finitely ramified fractals, named p.c.f. self-similar sets, including the Sierpinski gasket as a typical example, where Laplacians are defined as renormalized limits of graph Laplacians, playing the role of differential operators of second order on manifolds.

The harmonic functions on fractal domains may have some different nature from the classical ones as a consequence of the finitely ramified topology of the fractals. For example, the Hexagasket and the Vicsek set [7] consume degenerate harmonic structures so that nonconstant harmonic functions vanishing locally on small cells exist. It seems that such phenomenon always happens on those fractals containing nonjunction inner vertices.

Recently, Tsougkas [8] gives a proof on the non-degeneracy of harmonic structures on the level-n Sierpinski Gasket $\mathcal{SG}_n$, $n \geq 2$, based on certain geometric graph theory, in

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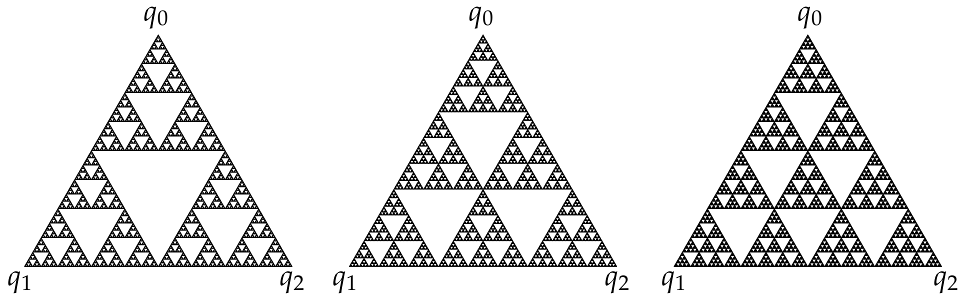

Figure 1: $\mathcal{SG}_2$, $\mathcal{SG}_3$, $\mathcal{SG}_4$.

which Tutte's spring theorem plays the key role, which was conjectured and numerically checked for the standard case for $n \leq 50$ by Hino [1,2].

In this note, we aim to provide an elementary proof for this theorem using only the basic facts of the discrete Laplacians on finite sets.

Theorem 1.1. *For $n \geq 2$, each harmonic structure of $\mathcal{SG}_n$ is non-degenerate.*

For convenience, we list some basic concepts and facts on the Laplacians on finite sets below. Readers can find any unexplained details in the book [5].

Definition 1.1. Let V be a finite set. A symmetric linear operator (matrix) $H : l(V) \to l(V)$ is called a Laplacian on V if it satisfies

(1) H is non-positive definite,

(2) $Hu = 0$ if and only if u is a constant on V,

(3) $H_{pq} \geq 0$ for all $p \neq q$.

Denote $\mathcal{LA}(V)$ the set of Laplacians on V.

Recall that $\mathcal{SG}_n$ is the unique nonempty compact subset of $\mathbb{R}^2$ satisfying

$$\mathcal{SG}_n = \bigcup_{i=0}^{\frac{n^2+n-2}{2}} F_i \mathcal{SG}_n$$

with F_i's being contraction mappings defined as $F_i(z) = n^{-1}z + d_{n,i}$ with suitable $d_{n,i} \in \mathbb{R}^2$. See Fig. 1. The set V_0 consisting of the three vertices q_0, q_1, q_2 of the smallest triangle containing $\mathcal{SG}_n$ is called the boundary. In this note, we mainly discuss the Laplacians on

$$V_1 = \bigcup_{i=0}^{\frac{n^2+n-2}{2}} V_0.$$