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Singular Functions and Characterizations of Field Concentrations: a Survey

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In the presence of closely located inclusions of the extreme material property, the physical fields, such as the electric field and the stress tensor, may be concentrated and arbitrarily large in the narrow region between two inclusions. Recently there has been significant progress on the quantitative characterization of the field concentration in the contexts of electrostatics (Laplace equation), linear elasticity (Lamé system), and viscous flow (Stokes system). This paper is to review such progress in a coherent way.

Key Words: Field concentration, gradient blow-up, closely spaced inclusions, extreme inclusions, Laplace equation, Lamé system, Stokes system.

AMS Subject Classifications: 35J47, 35B40, 35B45

1 Introduction

When two inclusions of the extreme material property are located closely to each other, the physical field may be concentrated and arbitrarily large in the narrow region between the inclusions. An inclusion of the extreme material property means a perfectly conducting or insulating inclusion (the conductivity being ∞ or 0) in the electrostatic case and a hard inclusion and a hole in the elastostatic case, and the corresponding physical fields are the electric field and the stress tensor. Such field concentration may occur in fiber-reinforced composites causing failure of the composites [6], and the electric field can be greatly enhanced and utilized to achieve subwavelength imaging and sensitive spectroscopy [35]. In this respect it is quite important to understand the field concentration in a quantitatively precise manner. It is also quite important to come up with an efficient

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numerical scheme to compute the fields in such cases since numerical computation of the field is known to be very hard in the presence of closely located inclusions.

In response to such importance and mathematical challenges involved in this problem, there has been much progress in understanding the field concentration in the last 20 years or more. In the context of electrostatics (or anti-plane elasticity in two dimensions), the field is the gradient of a solution to the Laplace equation and the precise estimates of the gradient were obtained when the conductivity of the inclusions is ∞: the blow-up rate of the gradient is $\epsilon^{-1/2}$ in two dimensions [5,37], where ϵ is the distance between two inclusions, and it is $|\epsilon \ln \epsilon|^{-1}$ in three dimensions [7]. There is a long list of literature in this direction of research among which we mention [3,4,9,13,21,27,28,33,34,38]. We also mention for related works [10,12,14,22–24]. If the conductivity of the inclusions is 0 (the insulating case), the two-dimensional problem is dual to the perfectly conducting case (by means of the harmonic conjugation), and hence the blow-up rate of the insulating case is also $\epsilon^{-1/2}$. But the three-dimensional case requires further investigation. In this respect, we mention the paper of Yun [39] where a rather unexpected blow-up rate of the gradient has been found when the inclusions are balls. If the conductivity is away from ∞ and 0, then the gradient stays bounded no matter how closely located the inclusions are [11,30,31].

While most of the work mentioned above focus on the estimates from above and below of the blow-up rate of the gradient, there is another important direction of research which is to characterize the singular behavior of the gradient. The characterization of the singular behavior means, roughly speaking, the decomposition of the solution u into the form $u = s + b$ where s carries the information of singularity of the gradient ∇u and b is a regular function in the sense that ∇b is bounded (or less singular) regardless of the distance between two inclusions. One important feature of such decompositions is that the singular part s is explicitly given and satisfies the governing equation, e.g., the conductivity equation, the elasticity equation, and so on. It has a significant implication on the numerical computation of the solution in presence of closely located inclusions. Such a computation is known to be a difficult problem because very fine meshes are required since the gradient becomes arbitrarily large in the narrow region. The decomposition enables us to compute the solution u numerically using standard meshes, not refined ones since s is explicit and b is regular. Such a characterization is reminiscent of that related to the corner singularity of elliptic equations which are utilized for computation of the solution to the (interior or exterior) boundary value problem when the domain has a corner [15,25,26].

Characterizations of the field concentration are obtained for the conductivity equation in [1,16–18,32] and for the Lamé system of the linear elasticity in two dimensions in [19] when inclusions are locally strictly convex. These result has been further extended to the two-dimensional Stokes system for circular inclusions [2]. The singular parts of the decomposition are represented by explicit building blocks, which we call singular functions. It is the purpose of this paper to summarize these results on the singularity characterization in a coherent way.