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Gradient Estimates of Solutions to the Conductivity Problem with Flatter Insulators

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. We study the insulated conductivity problem with inclusions embedded in a bounded domain in $\mathbb{R}^n$. When the distance of inclusions, denoted by ε, goes to 0, the gradient of solutions may blow up. When two inclusions are strictly convex, it was known that an upper bound of the blow-up rate is of order $\varepsilon^{-1/2}$ for $n = 2$, and is of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$ when dimension $n \geq 3$. In this paper, we generalize the above results for insulators with flatter boundaries near touching points.

Key Words: Conductivity problem, harmonic functions, maximum principle, gradient estimates.

AMS Subject Classifications: 35B44, 35J25, 35J57, 74B05, 74G70, 78A48

1 Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and let D_1^* and D_2^* be two open sets whose closure belongs to Ω, touching only at the origin with the inner normal vector of ∂D_1^* pointing in the positive x_n-direction. Denote $x = (x', x_n)$. Translating D_1^* and D_2^* by $\frac{\varepsilon}{2}$ along x_n-axis, we obtain

$$D_1^\varepsilon := D_1^* + \big(0', \varepsilon/2\big) \quad \text{and} \quad D_2^\varepsilon := D_2^* - \big(0', \varepsilon/2\big).$$

When there is no confusion, we drop the superscripts ε and denote $D_1 := D_1^\varepsilon$ and $D_2 := D_2^\varepsilon$. Denote $\widetilde{\Omega} := \Omega \setminus \overline{(D_1 \cup D_2)}$. A simple model for electric conduction can be formulated as the following elliptic equation:

$$\begin{cases} \operatorname{div}\Big(a_k(x)\nabla u_k\Big) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0,\infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \widetilde{\Omega}, \end{cases}$$

refers to conductivities. The solution u_k and its gradient ∇u_k represent the voltage potential and the electric fields respectively. From an engineering point of view, It is an interesting problem to capture the behavior of ∇u_k. Babuška, et al. [3] numerically analyzed that the gradient of solutions to an analogous elliptic system stays bounded regardless of ε, the distance between the inclusions. Bonnetier and Vogelius [5] proved that for a fixed k, $|\nabla u_k|$ is bounded for touching disks D_1 and D_2 in dimension $n = 2$. A general result was obtained by Li and Vogelius [11] for general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions D_1 and D_2 in any dimension. When k is bounded away from 0 and ∞, they established a $W^{1,\infty}$ bound of u_k in Ω, and a $C^{1,\alpha}$ bound in each region that do not depend on ε. This result was further extended by Li and Nirenberg [10] to general second order elliptic systems of divergence form. Some higher order estimates with explicit dependence on r_1, r_2, k and ε were obtained by Dong and Li [7] for two circular inclusions of radius r_1 and r_2 respectively in dimension $n = 2$. There are still some related open problems on general elliptic equations and systems. We refer to p. 94 of [11] and p. 894 of [10].

When the inclusions are insulators ($k = 0$), it was shown in [6,9,13] that the gradient of solutions generally becomes unbounded, as $\varepsilon \to 0$. It was known that (see e.g., Appendix of [4]) when $k \to 0$, u_k converges to the solution of the following insulated conductivity problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \quad i = 1,2, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here ν denotes the inward unit normal vectors on ∂D_i, $i = 1,2$.

The behavior of the gradient in terms of ε has been studied by Ammari et al. in [1] and [2], where they considered the insulated problem on the whole Euclidean space:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{(D_1 \cup D_2)}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \quad i = 1,2, \\ u(x) - H(x) = \mathcal{O}(|x|^{n-1}) & \text{as } |x| \to \infty. \end{cases} \tag{1.3}$$

They established when dimension $n = 2$, D_1^* and D_2^* are disks of radius r_1 and r_2 respectively, and H is a harmonic function in $\mathbb{R}^2$, the solution u of (1.3) satisfies

$$\|\nabla u\|_{L^\infty(B_4)} \le C\varepsilon^{-1/2},$$