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Keplerian Action, Convexity Optimization, and the 4-Body Problem

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper we introduce a method to construct periodic solutions for the n-body problem with only boundary and topological constraints. Our approach is based on some novel features of the Keplerian action functional, constraint convex optimization techniques, and variational methods. We demonstrate the strength of this method by constructing relative periodic solutions for the planar four-body problem within a special topological class, and our results hold for an open set of masses.

Key Words: n-body problem, variational methods, periodic solutions, convex optimization.

AMS Subject Classifications: 70F10, 37J51

1 Introduction

The Newtonian n-body body problem concerns the motion of n masses $m_1, \cdots, m_n \geq 0$ moving in $\mathbb{R}^d$, $d \in \{1,2,3\}$, in accordance with Newton's law of universal gravitation:

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \quad k = 1, \cdots, n, \tag{1.1}$$

where $x_k \in \mathbb{R}^d$ is the position of m_k, $x = (x_1, \cdots, x_n)$, and

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|}$$

is the (self-)potential energy. Let

$$K(\dot{x}) = \frac{1}{2} \sum_{k=1}^{n} m_k |\dot{x}_k|^2$$

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be the kinetic energy and $L(x,\dot{x}) = U(x) + K(\dot{x})$ be the Lagrangian. Eq. (1.1) are Euler-Lagrange equations for the action functional

$$\mathcal{A}_{t_0,t_1}(x) = \int_{t_0}^{t_1} L(x,\dot{x})dt, \quad x \in H^1_{\rm loc}(\mathbb{R},\mathbb{C}^n). \tag{1.2}$$

The case $\mathcal{A}_{0,T}$ will be denoted by $\mathcal{A}_T$. Unless specified otherwise, throughout this paper a "solution" of (1.1) is referred to a "classical solution" of (1.1).

Analytic construction for periodic solutions of (1.1) is an old school, while variational approach has become a fashion since the discovery of the hip-hop orbit with four bodies [16] and the figure-8 orbit [15] with three bodies. Their idea of imposing symmetry constraints on solution curves was subsequently applied to many other examples, some notable successes been choreographic solutions [3,7,13,14,21,22,25,28,29], multiple choreographic solutions (such as the parallelogram four-body problem) [5–7], generalized hip-hops [12, Section 4.2] and [30], and many other orbits with miscellaneous types of symmetries (such as symmetries with rotating circle property) [4, 17–19]. Most applications rely on manipulations of some equal masses. There are some examples without restriction on equal masses: the generalized hip-hops with the Italian symmetry [12, Section 4.2], some Hill type orbits [2], retrograde orbits for the three-body problem [9, 10], and certain orbits with n-bodies extending Euler-Moulton relative equilibria [11]. In some of these examples, simple order-two spatial symmetry were imposed without involving permutation of masses. Apart from them, to our knowledge there seems to be no substantial progress on variational constructions for periodic solutions of (1.1) with totally distinct masses.

Numerical experiments suggest that, however, many highly symmetric orbits with identical masses persist as one perturb the masses, with the only expense being the lose of some symmetry. The persistence is in fact observed in many examples for a fairly large range of masses. Some curious experiments on perturbing masses for orbits in [5] and [7, Section 5] are major incentives of our present work. Fig. 1 is a very small list of motivating examples. With totally distinct masses, manipulations with symmetries are not helpful. Direct applications of global estimates in [9, 10] are also not quite useful for n-body problems with $n \geq 4$, as to be explained later in this paper (Section 4). These solutions fall in certain topological families, and it is in general a difficult task to rigorously prove the existence of a real solution within a given topological family of curves. There must be some insights and artifices missing.

The purpose of this paper is to introduce a method to construct periodic solutions for the n-body problem with only boundary and topological constraints (Section 3). Our approach is based on some novel features of the Keplerian action functional (Section 2), some properties of the action functional (Section 4), and some constraint convex optimization techniques (Section 5). Our approach is a substantial improvement of methods in [9,10], and has no restriction on equal masses. We illustrate the strength of this method by constructing relative periodic solutions for the planar four-body problem within a special topological class (Section 6).