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Ground States to the Generalized Nonlinear Schrödinger Equations with Bernstein Symbols

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. This paper concerns with existence and qualitative properties of ground states to generalized nonlinear Schrödinger equations (gNLS) with abstract symbols. Under some structural assumptions on the symbol, we prove a ground state exists and it satisfies several fundamental properties that the ground state to the standard NLS enjoys. Furthermore, by imposing additional assumptions, we construct, in small mass case, a nontrivial radially symmetric solution to gNLS with H^1-subcritical nonlinearity, even if the natural energy space does not control the H^1-subcritical nonlinearity.

Key Words: Generalized NLS, solitary waves, variational methods, Bernstein symbols.

AMS Subject Classifications: 35J35, 35J61

1 Introduction

We consider the generalized nonlinear Schrödinger equation (NLS for abbreviation) of the form

$$i\partial_t \psi = P(-\Delta)\psi - |\psi|^{p-1}\psi, \tag{1.1}$$

where $\psi = \psi(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ with $d \geq 2$ and $P(-\Delta)$ is defined as a Fourier multiplier of symbol $P = P(\lambda) : [0,\infty) \to \mathbb{R}$. The nonlinear Schrödinger equation is a universal equation describing dynamics of waves in various physical contexts. When $P(\lambda) = \lambda$, the equation is simply the standard NLS, and it appears in nonlinear optics or as a mean-field approximation to the many-body bosonic system The case $P = \lambda^{s/2}, 0 < s < 2$ arises in the study of fractional Schrödinger equations [15]. When relativistic effects are taken in account, the symbol $P = \sqrt{\lambda + m^2} - m$, or more generally $P = (\lambda + m^2)^{s/2} - m^s$ with $0 < s < 2$, is chosen [18].

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In this article, we are concerned with the ground state solution to the generalized NLS (1.1). By a ground state, we mean a standing wave solution of the form

$$\psi(t,x) = e^{i\mu t}u(x), \quad \mu > 0, \tag{1.2}$$

which minimizes the value of the action integral. A rigorous definition of a ground state shall be given in Section 4. As for the standard NLS, i.e., the case $P(\lambda) = \lambda$, inserting the standing wave ansatz (1.2) into (1.1), we get the standard stationary NLS

$$-\Delta u + \mu u = |u|^{p-1}u. \tag{1.3}$$

In this case, the theory of ground states has been almost completed during several decades. A criteria for their existence and nonexistence, depending on the range of μ and p, is established in [1,22,23]. Qualitative properties of ground states, such as positiveness, radial symmetry, monotonicity, and uniqueness have been proved in [4,13,14].

With a general symbol P, the stationary generalized NLS is given by

$$P(-\Delta)u + \mu u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^d. \tag{1.4}$$

For some special choices of P, such as $\lambda^{s/2}$ and $\sqrt{\lambda + m^2} - m$, a great deal of intensive works on ground states to (1.4) have been carried out. An important remark is that ground states to (1.4) with aforementioned symbols share common qualitative properties such as sign-definiteness, radial symmetry, monotone decreasing property and uniqueness [3,5,6,9–11,16,20].

In this paper, we are interested in finding general conditions on the symbol P which allow ground states to the generalized NLS (1.4) to have the same kinds of qualitative properties. We propose the following structural assumptions for the symbol P:

(H1) $P : [0,\infty) \to [0,\infty)$ is continuous on $[0,\infty)$ and smooth on $(0,\infty)$;

(H2) P is a Bernstein function, i.e., P' is totally monotone (see Section 2 for definition);

(H3) there exists $s \in (0,2]$ such that $P(\lambda) \gtrsim \lambda^{\frac{s}{2}}$ for all large λ.

Important examples of differential operators satisfying the assumptions (H1)–(H3) include the factional Laplacians $P(\lambda) = \lambda^{s/2}$ with $0 < s < 2$ and the generalized pseudo-relativistic operators $P = (\lambda + m^2)^{s/2} - m^s$ with $0 < s < 2$. Some algebraic functions $\frac{\lambda}{(\lambda+1)^s}$, $0 < s < 1$ or $\frac{\lambda^\beta - 1}{\lambda^\alpha - 1} - 1$, $0 < \alpha < \beta < 1$ are also included. For more examples satisfying (H1)–(H3), we refer to the comprehensive book [21].

Our first theorem states that by assuming (H1)–(H3), one can construct a ground state to (1.4) that fulfills desired qualitative properties.

Theorem 1.1 (Existence of a ground state)**.** *Suppose* (H1)–(H3). *Let* $p \in (1, (d+s)/(d-s))$ *be given. Then for any* $\mu > 0$, *the generalized* NLS

$$P(-\Delta)u + \mu u = |u|^{p-1}u \quad \mathit{in} \quad \mathbb{R}^d \tag{1.5}$$

possesses a ground state $u \in H_{P+\mu}$ *which is positive, radially symmetric and monotone decreasing in the radial direction.*