Analysis in Theory and Applications
Anal. Theory Appl., Vol. **37**, No. 2 (2021), pp. 178-190

DOI: 10.4208/ata.2021.pr80.01
June 2021

Diffusion with a Discontinuous Potential: a Non-Linear Semigroup Approach

Yong-Jung Kim[1,*] and Marshall Slemrod[2]

[1] *Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 305-701, Korea*
[2] *Department of Mathematics, University of Wisconsin, Madison WI 53706, USA*

Received 31 May 2020; Accepted (in revised version) 23 July 2020

Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. This paper studies existence of mild solution to a sharp cut off model for contact driven tumor growth. Analysis is based on application of the Crandall-Liggett theorem for ω-quasi-contractive semigroups on the Banach space $L^1(\Omega)$. Furthermore, numerical computations are provided which compare the sharp cut off model with the tumor growth model of Perthame, Quirós, and Vázquez [13].

Key Words: Nonlinear semigroups, tumor growth models, Hele-Shaw diffusion.

AMS Subject Classifications: 35Q92, 92C50

1 Introduction

In their paper [13], Perthame, Quirós, and Vázquez proposed the following model for tumor growth

$$v_t = \nabla \cdot (v \nabla p) + v(1-p), \quad v(0) = v_0, \tag{1.1}$$

where v is the cell density, v_0 is the initial value, and p is the pressure field. In their model, the pressure field is approximated by

$$p \cong p_m := \frac{m}{1-m} \Big(\frac{v}{v_c}\Big)^{m-1}, \tag{1.2}$$

where the coefficient v_c is the maximum packing density and is set to $v_c = 1$ for convenience. In this case, Eq. (1.1) is written as

$$v_t = \Delta v^m + v(1-p_m). \tag{1.3}$$

*Corresponding author. *Email addresses:* `yongkim@kaist.edu` (Y-J. Kim), `slemrod@math.wisc.edu` (M. Slemrod)

The contact driven tumor growth model is taken as limit $m \to \infty$ of (1.3). Perthame, Quirós, and Vázquez [13] proved that, if the initial value is smooth and bounded

$$0 \le v_0 \le 1, \tag{1.4}$$

the pair (v_m, p_m) converges to (v_∞, p_∞) as $m \to \infty$ which satisfy a Hele-Shaw type diffusion model

$$\partial_t v_\infty = \Delta p_\infty + v_\infty(1 - p_\infty), \quad v_\infty(0) = v_0, \tag{1.5}$$

in the sense of distributions. Furthermore,

$$v_\infty \in C((0,\infty); L^1(\mathbb{R}^N) \cap BV(\mathbb{R}^N \times (0,\infty)), \quad 0 \le v_\infty \le 1,$$

and

$$p_\infty \in P_\infty(v_\infty), \quad 0 \le p_\infty \le 1, \tag{1.6}$$

where the inclusion relation (1.6) is given by the set-valued function

$$P_\infty(v) = \begin{cases} 0, & 0 \le v < 1, \\ [0,\infty), & v = 1. \end{cases} \tag{1.7}$$

Note that the diffusion term in (1.5) is present only when $v_\infty = 1$. Indeed, the limiting case gives an extreme scenario that the domain is divided into two parts, specifically when (a) the diffusion does not appear at all or (b) it is concentrated at $v = 1$.

We note the Hele-Shaw diffusion equation, (1.5)-(1.7), cannot be used as a model for the limiting case. First, it does not single out a solution (though to be fair the extended version of (3.1a) that we introduce in Section 3 will also be defined by an inclusion relation). The main reason is that the set-valued function $P_\infty(v)$ has discontinuity at the stable steady state of the reaction term, $v = 1$. Furthermore, if the initial value is not bounded by (1.4), the solution is not defined. As an alternative system, we consider a sharp cut off model

$$u_t = \Delta G(u) + f(u), \quad u(0) = u_0 \ge 0, \tag{1.8}$$

where

$$G(u) = \begin{cases} 0, & u < 1, \\ 1, & u \ge 1, \end{cases} \tag{1.9a}$$

$$f(u) = \begin{cases} u, & 0 \le u < 1, \\ 0, & u \ge 1, \end{cases} \tag{1.9b}$$

which has been introduced by Kim and Pan [11]. We set $G(1) = 1$ in (1.9a) to connect the model to the nonlinear diffusion in (1.3) which has the same property, i.e., $v^m = 1$ when $v = 1$. The value of the potential at the discontinuity point $u = 1$ makes a difference since it is a stable steady state of the reaction function $f(u)$.