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Deformation Argument under PSP Condition and Applications

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper we introduce a new deformation argument, in which C^0-group action and a new type of Palais-Smale condition PSP play important roles. This type of deformation results are studied in [17, 21] and has many different applications [10, 11, 17, 21] et al. Typically it can be applied to nonlinear scalar field equations. We give a survey in an abstract functional setting. We also present another application to nonlinear elliptic problems in strip-like domains. Under conditions related to [5,6], we show the existence of infinitely many solutions. This extends the results in [8].

Key Words: Deformation theory, nonlinear elliptic equations, radially symmetric solutions, strip-like domains, Pohozaev functional.
AMS Subject Classifications: 35B38, 35J20, 58E05

1 Introduction

We study nonlinear differential equations with scaling properties via variational methods. A typical example is the following nonlinear scalar field equations:

$$-\Delta u = g(u) \quad \text{in } \mathbf{R}^N, \tag{1.1a}$$

$$u \in H^1(\mathbf{R}^N), \tag{1.1b}$$

where $N \geq 2$ and we consider the existence of radially symmetric solutions. This type of problem appears in many models in mathematical physics and is well-studied by many authors. Especially Berestycki and Lions [5, 6] and Berestycki, Gallouët and Kavian [7] obtained almost necessary and sufficient conditions for the existence of non-trivial solutions. More precisely they consider (1.1) under the following conditions

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(g0) $g(\xi) \in C(\mathbf{R}, \mathbf{R})$ and $g(\xi)$ is odd.

(g1) For $N \geq 3$,

$$\limsup_{\xi \to \infty} \frac{g(\xi)}{\xi^{(N+2)/(N-2)}} \leq 0.$$

For $N = 2$,

$$\limsup_{\xi \to \infty} \frac{g(\xi)}{e^{\alpha \xi^2}} \leq 0 \quad \text{for any} \quad \alpha > 0.$$

(g2) $-\infty < \liminf_{\xi \to 0} \frac{g(\xi)}{\xi} \leq \limsup_{\xi \to 0} \frac{g(\xi)}{\xi} < 0.$

(g3) There exists a $\zeta_0 > 0$ such that $G(\zeta_0) > 0$, where $G(\xi) = \int_0^\xi g(\tau)\, d\tau$.

The [5,6] (for $N \geq 3$) and [7] (for $N = 2$) showed the existence of a positive solution of (1.1) and infinitely many possibly sign-changing radially symmetric solutions. We note that in [5–7] solutions are found as critical points of constraint functional

$$u \mapsto \int_{\mathbf{R}^N} |\nabla u|^2\, dx;\ \Big\{ u \in H_r^1(\mathbf{R}^N);\ \int_{\mathbf{R}^N} G(u)\, dx = 1 \Big\} \to \mathbf{R} \qquad \text{for } N \geq 3, \tag{1.2a}$$

$$u \mapsto \int_{\mathbf{R}^2} |\nabla u|^2\, dx;\ \Big\{ u \in H_r^1(\mathbf{R}^2);\ \int_{\mathbf{R}^2} G(u)\, dx = 0 \Big\} \to \mathbf{R} \qquad \text{for } N = 2, \tag{1.2b}$$

after a suitable scaling and solutions satisfy Pohozaev identity. See Coleman, Glazer and Martin [12] for related argument. We also note that a positive solution is obtained as a minimizer after scaling and it is a least energy solution.

Remark 1.1. When $N = 2$, in [7] the existence of solution is obtained under slightly stronger conditions (g0), (g1), (g3) and

(g2′) $\lim_{\xi \to 0} \frac{g(\xi)}{\xi} < 0$ exists.

In [16], Hirata, Ikoma and the second author introduced a new approach to (1.1), in which we try to apply minimax argument to the natural functional associated to (1.1):

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2\, dx - \int_{\mathbf{R}^N} G(u)\, dx :\ H_r^1(\mathbf{R}^N) \to \mathbf{R}. \tag{1.3}$$

We note that it is difficult to verify so-called Palais-Smale condition ((PS) in short) for $I(u)$ and we cannot apply the standard deformation argument directly to $I(u)$. We also remark that the constraint functional (1.2a) and (1.2b) satisfy (PS) condition.

To avoid lack of (PS) condition, we make use of a special scaling property of the functional $I(u)$ and we introduce following Pohozaev functional:

$$P(u) = \frac{N-2}{2} \int_{\mathbf{R}^N} |\nabla u|^2\, dx - N \int_{\mathbf{R}^N} G(u)\, dx :\ H_r^1(\mathbf{R}^N) \to \mathbf{R}.$$