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Multiple Sign-Changing Solutions for Quasilinear Equations of Bounded Quasilinearity

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. The existence of an infinite sequence of sign-changing solutions are proved for a class of quasilinear elliptic equations under suitable conditions on the quasilinear coefficients and the nonlinearity

$$\begin{cases} \sum_{i,j=1}^{N} \left(b_{ij}(u) D_{ij} u + \frac{1}{2} D_z b_{ij}(u) D_i u D_j u \right) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and we use

$$D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \text{and} \quad D_z b_{ij}(z) = \frac{d}{dz} b_{ij}(z).$$

The main interest of this paper is for the case of bounded quasilinearity b_{ij}. The result is proved by an elliptic regularization method involving truncations of both u and the gradient of u.

Key Words: Quasilinear elliptic equations, sign-changing solution, an elliptic regularization method.

AMS Subject Classifications: 35J60, 35J20

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1 Introduction

In this paper, we study the existence of sign-changing solutions for the following quasilinear elliptic equation

$$\begin{cases} \sum_{i,j=1}^{N} \Big(b_{ij}(u) D_{ij} u + \frac{1}{2} D_z b_{ij}(u) D_i u D_j u \Big) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and we use the notations

$$D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad D_z b_{ij}(z) = \frac{d}{dz} b_{ij}(z).$$

We assume the following conditions on b_{ij} and f. Denote the critical exponent by $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$.

(b_1) Let $b_{ij} = b_{ji} \in C^{1,1}(\mathbb{R}, \mathbb{R})$ for $i, j = 1, \cdots, N$, satisfy that there exist positive constants b_-, b_+ such that

$$b_- |\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(z) \xi_i \xi_j \leq b_+ |\xi|^2 \quad \text{for } z \in \mathbb{R}, \quad \xi = (\xi_i) \in \mathbb{R}^N.$$

(b_2) There exist constants $q > 2$, $\delta > 0$ such that

$$\begin{aligned} \delta |\xi|^2 \leq & \sum_{i,j=1}^{N} \Big(b_{ij}(z) + \frac{1}{2} z D_z b_{ij}(z) \Big) \xi_i \xi_j \\ \leq & \frac{q}{2} \Big(\sum_{i,j=1}^{N} b_{ij}(z) \xi_i \xi_j - \delta |\xi|^2 \Big) \quad \text{for } z \in \mathbb{R}, \quad \xi \in \mathbb{R}^N. \end{aligned}$$

(b_3) There exists a positive constant c such that

$$|D_z b_{ij}(z) - D_z b_{ij}(w)| \leq c |z - w| \quad \text{for } z, w \in \mathbb{R}.$$

(b_4) $b_{ij}(z)$ is even in z.

(f_1) Let $f \in C(\mathbb{R}, \mathbb{R})$ satisfy that there exist constants $c > 0$ and $r \in (2, 2^*)$ such that

$$|f(z)| \leq c(1 + |z|^{r-1}) \quad \text{for } z \in \mathbb{R}.$$