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Classification of Positive Ground State Solutions with Different Morse Indices for Nonlinear N-Coupled Schrödinger System

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper, we study the following N-coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u_j + u_j = \mu_j u_j^3 + \sum\limits_{i\neq j} \beta_{i,j} u_i^2 u_j, & \text{in } \mathbb{R}^n, \\ u_j > 0 \quad \text{in } \mathbb{R}^n, \ \ u_j(x) \to 0 \quad \text{as } |x| \to +\infty, \ \ j = 1, \cdots, N, \end{cases}$$

where $n \leq 3$, $N \geq 3$, $\mu_j > 0$, $\beta_{i,j} = \beta_{j,i} > 0$ are constants and $\beta_{j,j} = \mu_j$, $j = 1, \cdots, N$. There have been intensive studies for the system on existence/non-existence and classification of ground state solutions when $N = 2$. However fewer results about the classification of ground state solution are available for $N \geq 3$. In this paper, we first give a complete classification result on ground state solutions with Morse indices 1, 2 or 3 for three-coupled Schrödinger system. Then we generalize our results to N-coupled Schrödinger system for ground state solutions with Morse indices 1 and N. We show that any positive ground state solutions with Morse index 1 or Morse index N must be the form of $(d_1 w, d_2 w, \cdots, d_N w)$ under suitable conditions, where w is the unique positive ground state solution of certain equation. Finally, we generalize our results to fractional N-coupled Schrödinger system.

Key Words: Nonlinear Schrödinger system, unique ground state solution, variational method, Morse indices.

AMS Subject Classifications: 35B09, 35J47, 35J50

1 Introduction

In this paper, we study the following N-coupled nonlinear Schrödinger system

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$$\begin{cases} -\Delta u_j + u_j = \mu_j u_j^3 + \sum\limits_{i\neq j} \beta_{i,j} u_i^2 u_j \quad \text{in } \mathbb{R}^n, \\ u_j > 0 \quad \text{in } \mathbb{R}^n, u_j(x) \to 0 \quad \text{as } |x| \to +\infty, \quad j = 1, \cdots, N, \end{cases} \tag{1.1}$$

where $n \leq 3$, $N \geq 3$, $\mu_j > 0$ are constants and $\beta_{i,j} = \beta_{j,i} > 0$ are coupling parameters ($\beta_{j,j} = \mu_j$). This paper is concerned with the uniqueness of ground state solution in the general case $N \geq 3$.

This system arises as standing wave solutions of the time-dependent N-coupled Schrödinger systems of the form

$$\begin{cases} -\sqrt{-1}\dfrac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j + \mu_j \Phi_j |\Phi_j|^2 + \sum\limits_{i\neq j} \beta_{i,j} \Phi_j^2 \Phi_j \quad \text{in } \mathbb{R}^n, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad t > 0, \quad j = 1, \cdots, N, \end{cases}$$

and these systems are also known as coupled Gross-Pitaevskii equations. In the past fifteen years, a great attention has been focused on the study of two coupled systems with nonlinear terms, both for their interesting theoretical structure and their concrete applications, such as in nonlinear optics and in Bose-Einstein condensates for multi-species condensates. By using variational methods or Lyapunov-Schmidt reduction methods, there are lots of results about existence, multiplicity and qualitative properties of nontrivial solutions of two coupled elliptic system. Since it seems almost impossible for us to provided a complete list of references, we refer the readers only to [1–11,18–20,25–27] and reference therein.

For two coupled Schrödinger system with $\beta_{1,2} = \beta_{2,1} = \beta$, B. Sirakov [24] showed that if $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, then $(\sqrt{k}w, \sqrt{l}w)$ is ground state solution, where k, l satisfies $\mu_1 k + \beta l = 1, \mu_1 l + \beta k = 1$ and they conjecture that under the above hypotheses $(\sqrt{k}w, \sqrt{l}w)$ is the unique positive solution. For this conjecture, by the ODE method, J. Wei and W. Yao, [27, Theorem 4.2] proved this conjecture in case $\beta > \max\{\mu_1, \mu_2\}$, and [27, Theorem 4.1] proved it in case $0 < \beta < \beta_1$, where β_1 is an unknown small constant. When $\beta < \min\{\mu_1, \mu_2\}$, Z. Chen and W. Zou [10] gave a complete answer to this conjecture and obtained the asymptotic behavior of ground state solution. However, the above work are for purely attractive and purely repulsive cases, there have been few results in the case of mixed couplings, i.e., the case having both positive and negative coupling constants. For the systems in the entire space with mixed couplings was considered by T. Lin and J. Wei [16], in which a 3-system was considered with two coupling constants positive and one coupling constant negative.

For N-coupled system with mixed couplings, J. Wei and T. Lin [16] established some general theorems for the existence and nonexistence of ground state solution and showed that when all $\beta_{i,j}$ are positive and the matrix $\mathcal{B}$ is positively definite, there exist a ground state solution which is radially symmetric. However, if all $\beta_{i,j}$ are negative, or one of $\beta_{i,j}$ is negative and the matrix $\mathcal{B}$ is positively definite, there is no ground state solution. Recently, J. Wei and Y. Wu [28] gave a systematic and an (almost) complete study on