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Littlewood-Paley Functions and Triebel-Lizorkin Spaces, Besov Spaces

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. We establish Littlewood-Paley charaterizations of Triebel-Lizorkin spaces and Besov spaces in Euclidean spaces using several square functions defined via the spherical average, the ball average, the Bochner-Riesz means and some other well-known operators. We provide a simple proof so that we are able to extend and improve many results published in recent papers.

Key Words: Littlewood-Paley square functions, Triebel-Lizorkin spaces, Besov spaces, spherical average, ball average, Bochner-Riesz means.

AMS Subject Classifications: 46E35, 42B25, 42B35

1 Introduction

As is well known, Littlewood-Paley functions and their various applications are important parts of harmonic analysis, dating back as far as the early 1930's; see [10, 19, 20, 22] for more details about the historical development. Recently Alabern et al. in [1] obtained a new characterization of Sobolev spaces with arbitrary smoothness order on Euclidean spaces, which can be seen as characterizations of Sobolev spaces via Littlewood-Paley g-functions involving ball averages. These characterizations depend only on the metric of $\mathbb{R}^n$ and hence provide several possible approaches to introduce high order Sobolev spaces on general metric measure spaces. Motivated by the work in [1], some characterizations of high order Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$ in terms of Littlewood-Paley functions and pointwise inequalities involving ball averages were further established, which also serve as new approaches to introduce these spaces with high order smoothness on metric measure spaces. Yang et al. in [26] established the corresponding characterizations for Besov and Triebel-Lizorkin spaces. Inspired by [1, 26],

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Dai et al. further characterized Sobolev spaces with any positive even integer order via some pointwise inequalities involving ball averages in [6], as well as Besov and Triebel-Lizorkin spaces with any positive smoothness order via some Littlewood-Paley g functions involving ball averages in [7]. Based on [7], Chang et al. in [3] considered the related characterizations of Triebel-Lizorkin spaces via the corresponding Lusin area function and the Littlewood-Paley g_λ^* -function. Some further characterizations of Sobolev, Besov and Triebel-Lizorkin spaces via ball averages were then presented in a series of works [4, 8, 13, 15, 25, 28–30].

One the other hand, Chen et al. in [5] gave a simple method to characterize inhomogeneous Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ by using several different square functions defined via the spherical average, the ball average and the Bochner-Riesz means. Based on the aforementioned works, the main purpose of this article is to characterize Triebel-Lizorkin and Besov spaces via some generalized Littlewood-Paley functions which are much more general than those Littlewood-Paley functions of ball averages. We extend their results, using an alternate, less complicated method of proof.

To this end, we firstly give some necessary notations. Let $n \geq 2$ and $\mathbb{R}^n$ be n-dimensional Euclidean space. Fix an $L^1(\mathbb{R}^n)$ function Φ. Denote, for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$,

$$\Phi_{2^t}(x) = 2^{-tn}\Phi\left(x/2^t\right).$$

The Fourier transform of Φ_{2^t} is given by $\widehat{\Phi_{2^t}}(\xi) = \widehat{\Phi}\left(2^t\xi\right)$, $\xi \in \mathbb{R}^n$. For any $1 < q < \infty$, associated with Φ, the Littlewood-Paley function $S_{\Phi,q}(f)$ is defined by

$$S_{\Phi,q}(f)(x) = \left(\int_{\mathbb{R}} |\Phi_{2^t} * f(x)|^q\, dt\right)^{1/q},$$

and the discrete version is given by

$$D_{\Phi,q}(f)(x) = \left(\sum_{k\in\mathbb{Z}} |(\Phi_{2^k} * f)(x)|^q\right)^{1/q}. \tag{1.1}$$

Sometimes we write $S_{\Phi,q}(f)(x)$ in an equivalent form

$$S_{\Phi,q}(f)(x) = \left(\int_0^\infty |\Phi_t * f(x)|^q \frac{dt}{t}\right)^{1/q},$$

and skip the ratio $(\ln 2)^{-1/q}$ between two forms. Also, for simplicity, we initially define $S_{\Phi,q}(f)$ on all functions f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Let

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$