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Weighted ℓ_p-Minimization for Sparse Signal Recovery under Arbitrary Support Prior

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. Weighted ℓ_p $(0 < p \leq 1)$ minimization has been extensively studied as an effective way to reconstruct a sparse signal from compressively sampled measurements when some prior support information of the signal is available. In this paper, we consider the recovery guarantees of k-sparse signals via the weighted ℓ_p $(0 < p \leq 1)$ minimization when arbitrarily many support priors are given. Our analysis enables an extension to existing works that assume only a single support prior is used.

Key Words: Adaptive recovery, compressed sensing, weighted ℓ_p minimization, sparse representation, restricted isometry property.

AMS Subject Classifications: 90C26, 90C30, 94A20

1 Introduction

Compressed sensing [2, 5] is a new data acquisition paradigm, which reliably recovers a high dimensional sparse signal $\boldsymbol{x} \in \mathbb{R}^n$ (a signal is called k-sparse if the number of its nonzero entries has at most $k \ll n$) from significantly fewer linear observations

$$\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{e}, \tag{1.1}$$

where $\boldsymbol{\Phi} \in \mathbb{R}^{m\times n}$ is a measurement matrix and $\boldsymbol{e} \in \mathbb{R}^m$ denotes additive noise that satisfies $\|\boldsymbol{e}\|_2 \leq \epsilon$ for some known $\epsilon \geq 0$. Compressed sensing is nonadaptive because the measurement matrix $\boldsymbol{\Phi}$ does not depend on the signal being measured. But, some

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prior information of the signal x may be included in the estimates of the support of x or some estimates of largest coefficients of x in some settings. For example, video and audio signals exhibit strong correlation over temporal frames, which can be used to estimate a portion of the support based on previously decoded frames (see [6]). Therefore, the recovery of the signal x incorporating prior support information has received much attention including the weighted ℓ_1-minimization [3,4,6,14,16,17,19], the weighted ℓ_p $(0 < p < 1)$-minimization [10,11,13,18] and the greedy algorithm with partial support information [7,12,15].

This paper considers the recovery of the signal x from (1.1) and is devoted to new RIP bounds for the exact and stable recovery of sparse signals with arbitrary many support priors via the weighted ℓ_p-minimization:

$$\min_{x\in\mathbb{R}^n} \|x\|_{p,\mathbf{w}}^p \quad \text{subject to} \quad \|\Phi x - y\|_2 \le \varepsilon, \tag{1.2}$$

where $\mathbf{w} \in [0,1]^n$ is a weight vector and

$$\|x\|_{p,\mathbf{w}} = \Big(\sum_{i=1}^n \mathrm{w}_i |x_i|^p\Big)^{\frac{1}{p}}.$$

The main idea inherited in the weighted ℓ_p $(0 < p \le 1)$-minimization is to make the entries of x, which are "expected" to be large, be penalized less in the weighted objective function in (1.2) by the effect of the weight $\mathbf{w}$.

As $p = 1$, the method (1.2) reduces to the weighted ℓ_1-minimization:

$$\min_{x\in\mathbb{R}^n} \|x\|_{1,\mathbf{w}} \quad \text{subject to} \quad \|\Phi x - y\|_2 \le \varepsilon. \tag{1.3}$$

The rest of the paper is organized as follows. In Section 2, we recall a recently established RIP bound for signal recovery by virtue of the weighted ℓ_p-minimization with a single weight. In Section 3, we respectively present sufficient conditions for the recovery of sparse signals by weighted ℓ_p-minimization with non-uniform weights in both the noiseless and ℓ_2 bounded noise. Section 4 is devoted to the proofs of the main results.

2 Weighted ℓ_p-minimization with a single weight

Let $\widetilde{T} \subseteq [n] = \{1, 2, \cdots, n\}$ be a known single support estimate of x. The weight vector $\mathbf{w}$ in this case is taken by

$$\mathrm{w}_i = \begin{cases} \omega, & i \in \widetilde{T}, \\ 1, & i \in \widetilde{T}^c, \end{cases} \tag{2.1}$$

for some fixed $\omega \in [0,1]$ and $i \in [n]$.

The restricted isometry property (RIP) is one of the main tools used to evaluate the recovery performance via a variety of efficient algorithms. The RIP notion introduced by Candès et al. in [2], is the most widely used framework in compressed sensing.