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Gaussian BV Functions and Gaussian BV Capacity on Stratified Groups

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. Let G be a stratified Lie group and let $\{X_1,\cdots,X_{n_1}\}$ be a basis of the first layer of the Lie algebra of G. The sub-Laplacian Δ_G is defined by

$$\Delta_G=-\sum_{j=1}^{n_1}X_j^2.$$

The operator defined by

$$\Delta_G-\sum_{j=1}^{n_1}\frac{X_jp}{p}X_j$$

is called the Ornstein-Uhlenbeck operator on G, where p is a heat kernel at time 1 on G. In this paper, we investigate Gaussian BV functions and Gaussian BV capacities associated with the Ornstein-Uhlenbeck operator on the stratified Lie group.

Key Words: Gaussian p bounded variation, capacity, perimeter, stratified Lie group.

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1 Introduction

A function of bounded variation, simply a BV-function, is a real-valued function whose total variation is finite. In recent decades, many scholars have been paying attention to

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the BV function due to its application to calculus of variation and image processing. In the multi-variable setting, a function defined on an open subset $\Omega \subseteq \mathbb{R}^d, d \geq 2$, is said to have bounded variation provided that its distributional derivative is a vector-valued finite Radon measure over the subset Ω (cf. [2] or [13]). Precisely,

Definition 1.1. *A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation $\|Du\|$ in Ω is called a function of bounded variation, where*

$$\|Du\| := \sup\Big\{ \int_\Omega u \, div \, \nu dx: \ \nu = (\nu_1, \cdots, \nu_d) \in C_0^\infty(\Omega, \mathbb{R}^d), \ |\nu(x)| \leq 1, \ x \in \Omega \Big\} < \infty.$$

The class of all such functions will be denoted by $BV(\Omega)$. The norm of $BV(\Omega)$ is defined as

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + \|Du\|.$$

In the study of the pointwise behavior of a Sobolev function, the notion of capacity plays a crucial role. In recent years, the capacity related to bounded variation functions attracts the attentions of many researchers. Please refer to [13] for the classical BV-capacity in $\mathbb{R}^d$. After that, many scholars generalize the BV- capacity to other settings. Hakkarainen and Kinnunen [5] studied basic properties of the BV-capacity and the Sobolev capacity in a complete metric space equipped with a doubling measure and supporting a weak Poincaré inequality. In [11], J. Xiao introduced the BV-type capacity on Gaussian spaces $\mathbb{G}^d$, and as an application, the Gaussian BV-capacity was used to study the trace inequalities of Gaussian BV-space. Recently, the authors in this paper investigate the capacity and perimeters derived from α-Hermite Bounded Variation in [6]. The author in [9] investigates two analogues of the Ornstein-Uhlenbeck semi-group in the setting of stratified groups G, which can be regarded as the generalization of Gauss spaces to the case of Lie group. Motivated by the previous works, we will investigate the Gaussian BV function and the Gaussian BV capacity associated with Ornstein-Uhlenbeck operators on stratified Lie groups.

To state our results, we recall some basic facts on the stratified Lie group, which can be easily found in Folland and Stein's book [3]. Let G be a stratified group of dimension n with the Lie algebra $\mathfrak{g}$. This means that $\mathfrak{g}$ is equipped with a family of dilations $\{\alpha_r : r > 0\}$ and $\mathfrak{g}$ is a direct sum $\bigoplus_{j=1}^m \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\mathfrak{g}_1$ generates $\mathfrak{g}$, and $\alpha_r(X) = r^j X$ for $X \in \mathfrak{g}_j$. $Q = \sum_{j=1}^m j\, n_j$ is called the homogeneous dimension of G, where $n_j = \dim \mathfrak{g}_j$. G is topologically identified with $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \mapsto G$. α_r is also viewed as an automorphism of G and if $x \in G, r > 0$, we write

$$\alpha_r x = (r^{d_1} x_1, \cdots, r^{d_n} x_n),$$

where $1 \leq d_1 \leq \cdots \leq d_n$. We fix a homogeneous norm of G, which satisfies the generalized triangle inequalities

$$|xy| \leq \gamma(|x| + |y|) \qquad \text{for all } \ x, y \in G,$$

$$\big||xy| - |x|\big| \leq \gamma|y| \qquad \text{for all } \ x, y \in G \quad \text{with } \ |y| \leq \frac{|x|}{2},$$