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A Note on the Convergence of the Schrödinger Operator along Curve

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this paper we set up a convergence property for the fractional Schödinger operator for $0 < a < 1$. Moreover, we extend the known results to non-tangent convergence and the convergence along Lipschitz curves.

Key Words: Refinement of the Carleson problem, disconvergence set, fractional Schrödinger operator, Hausdorff dimension, Sobolev space.

AMS Subject Classifications: 42B25, 35Q20

1 Introduction

Given a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, we consider the fractional Schrödinger operator defined by

$$S_a(t)f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{ix\xi+it|\xi|^a} \hat{f}(\xi)d\xi \tag{1.1}$$

with $a > 0$. It is the solution to the initial data problem of the fractional Schrödinger equation

$$\begin{cases} \partial_t u(x,t) = (-\Delta)^{\frac{a}{2}} u(x,t), & \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

From the Plancherel theorem, (1.1) can be easily extend to a bounded operator on L^2-based Sobolev space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$. Here we recall the norm of $H^s(\mathbb{R}^n)$ as

$$\|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty. \tag{1.3}$$

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When $a=2$, $S_2(t)$ becomes the classical Schrödinger operator. We take $S(t)$ as its abbreviation. In [3], Carleson posed the following well known problem: To determine the infimum (critical) index s_c such that for any $s>s_c$,

$$\lim_{t\to 0} S(t)f(x)=f(x) \quad \text{a.e. } x\in\mathbb{R}^n, \quad \forall f\in H^s(\mathbb{R}^n). \tag{1.4}$$

For one dimensional case, Carleson [3] showed that (1.4) holds for $s\geq\frac{1}{4}$. The corresponding opposite result is obtained by Dahlberg and Kenig [7]. Moreover they showed that (1.4) does not hold for $s<\frac{1}{4}$ in any dimension. Thus we can conclude $s_c=1/4$ for $n=1$. After that, there are enumerate literatures devoted to settling the high dimensional problems. Sjölin [16] and Vega [20] proved the convergence if $s>1/2$ independently. Lee [11] set up (1.4) when $s>3/8$ and $n=2$. Bourgain [1] improved these results by showing that the convergence holds for $s>\frac{1}{2}-\frac{1}{4n}$ and the necessary condition is $s\geq\frac{1}{2}-\frac{1}{n}$ for $n\geq 4$. More recently, Bourgain [2] constructed a counter example to show that (1.4) does not hold for $s<\frac{n}{2(n+1)}$. Du, Guth and Li [6] obtained that $s_c=1/3$ by setting up (1.4) if $s>\frac{1}{3}$ and $n=2$. Forthermore, Du and Zhang [9] proved the convergence holds if $s>\frac{n}{2(n+1)}$ and $n\geq 3$. Thus the solution to Carleson's problem is $s_c=\frac{n}{2(n+1)}$ for $n\geq 2$.

It is nature to ask the same question for general $a>0$. An interesting phenomenon is that when $a>1$, the results do not depend on the value of a, but when $a<1$, the results depend on the value of it. For $a>1$, the convergence were proved to be true if $s>1/4$, $n=1$ by Sjölin [16] and Vega [20]. Miao, Yang, and Zheng [14] obtained the convergence when $s>\frac{3}{8}$ and $n=2$. Cho and Ko [4] proved that the convergence also holds when $s>\frac{n}{2(n+1)}$ and $n\geq 2$. The same result was also obtained by Li, Li and Xiao [12] by setting up the up-bound of Hausdorff dimension of the divergent set.

When $0<a<1$, Walther [21, 22] set up the convergence when $s>a/4$ in one dimension and for the radial functions in higher dimensional spaces. Very recently Dimou and Seeger [10] obtained the equivalent condition to time sequence of $\{t_n\}$ such that if $t_n\to 0$ (1.4) holds. Thus we know that $s_c=\frac{a}{4}$ is the critical index when $n=1$. For $n\geq 2$, Zhang [24] proved the convergence for $s>\frac{na}{4}$. It is still very open to determine the critical index for the high dimensional case.

An interesting generalization of the point-wise convergence problem is to set up the convergence in a wider approach region instead of vertical lines, for example, the non-tangential limit. It is easy to see that it holds for $s>\frac{n}{2}$ by Sobolev Embeding. Sjölin and Sjögren [15] showed that non-tangential convergence fails for $s\leq\frac{n}{2}$. Cho, Lee and Vargas [5] showed that the non-tangential convergence holds if $s>\frac{\beta(\Theta)+1}{4}$ when $a=2$ and $n=2$. $\beta(\Theta)$ denotes the upper Minkowski dimension of the upper cover of the cone which will be given soon. Cho, Lee and Vargas [5] deal with estimating the boundary of the operator along the restricted direction and time localization argument. Shiraki [17] extended result of [5] to $a>1$. In this paper, we will deal with the case of $0<a<1$, $n=1$.

To state our main results, we need first introduce in some notations. Let $\Theta\subset\mathbb{R}$ be a