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Compactness of the Commutators of Fractional Hardy Operator with Rough Kernel

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. The more explicit decomposition of the operator and the kernel are utilized to investigate a characterization of the central $BMO(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$ space via the compactness of the commutators of fractional Hardy operator with rough kernel.

Key Words: Fractional Hardy operator, commutator, compactness.

AMS Subject Classifications: 47B47, 47G10, 42B25

1 Introduction

Problem of commutators draws recently more and more attention of Harmonic analysis, such as its application in the study of elliptic equations [1,7]. For example, Sun, Wang and Zhang simplify the proof of the famous Wu's theorem on Navier-Stokes equations greatly in [18] and the technique used is some estimates for commutators by Lu and Yan [13]. The commutator formed by an operator T and a suitable function b can be recalled as

$$[b,T]f := b(Tf) - T(bf).$$

We call a function $b \in L_{loc}(\mathbb{R}^n)$ is a central $BMO(\mathbb{R}^n)$ (the mean oscillation function space) function, denoted by $CBMO(\mathbb{R}^n)$ which was introduced by Lu and Yang [14], if

$$\|b\|_{CBMO(\mathbb{R}^n)} := \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} |b(x) - b_{B_r}|dx < \infty.$$

Here and in what follows, $B_r := B(0,r)$ is a ball centered at 0 with radius $r > 0$. $CBMO(\mathbb{R}^n)$ can be understood as a local version of $BMO(\mathbb{R}^n)$ at the origin, $BMO(\mathbb{R}^n) \subset CBMO(\mathbb{R}^n)$ and they have quite different properties since for $1 < p < \infty$,

$$\|b\|_{BMO(\mathbb{R}^n)} \approx \|b\|_{BMO^p(\mathbb{R}^n)} \quad \text{and} \quad \|b\|_{CBMO(\mathbb{R}^n)} \lesssim \|b\|_{CBMO^p(\mathbb{R}^n)}$$

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with

$$\|b\|_{BMO^p(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B |b(x)-b_B|^p dx\right)^{\frac{1}{p}},$$
$$\|b\|_{CBMO^p(\mathbb{R}^n)} = \sup_{r>0}\left(\frac{1}{|B_r|}\int_{B_r} |b(x)-b_{B_r}|^p dx\right)^{\frac{1}{p}}.$$

Thus, the John-Nirenberg inequality is not true for $CBMO(\mathbb{R}^n)$. We follow the notation used in the existed work: $VMO(\mathbb{R}^n)$ denotes the $BMO(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$ (the space of all functions being infinite-times continuously differential in $\mathbb{R}^n$ with compact support), $CVMO(\mathbb{R}^n)$ stands for the $CBMO(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$.

This paper provides a characterization of the $CVMO(\mathbb{R}^n)$ space by the compactness of $[b,T]$, when T is the following fractional Hardy operator

$$H_{\Omega,\alpha}f(x) = \frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|}\Omega(x-y)f(y)dy,$$
$$H^*_{\Omega,\alpha}f(x) = \int_{|y|\geq|x|}\frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}}dy, \quad 0<\alpha<n.$$

Here Ω satisfies

$$\Omega(tx)=\Omega(x), \qquad \forall t>0, \quad x\in\mathbb{R}^n, \tag{1.1a}$$
$$\int_{\mathbb{S}^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.1b}$$
$$\Omega\in L^q(\mathbb{S}^{n-1}), \qquad \forall q\geq 1. \tag{1.1c}$$

The $L^{q\geq1}$-Dini condition of Ω can be recalled as

$$\int_0^1 \frac{w_q(\delta)}{\delta} < \infty \quad \text{with } w_q(\delta) = \sup_{\|\tau\|\leq\delta}\left(\int_{\mathbb{S}^{n-1}}|\Omega(\tau x')-\Omega(x')|^q d\sigma(x')\right)^{\frac{1}{q}}$$

and τ is a rotation on $\mathbb{S}^{n-1}$ with

$$\|\tau\| = \sup_{x'\in\mathbb{S}^{n-1}}|\tau x'-x'|.$$

For a suitable function h, $H^*_{\Omega,\alpha}$ is said to be the dual operator of $H_{\Omega,\alpha}$ in the following sense

$$\int_{\mathbb{R}^n} h(x)H_{\Omega,\alpha}f(x)dx = \int_{\mathbb{R}^n} f(x)H^*_{\Omega,\alpha}h(x)dx.$$

Fu, Lu and Zhao considered the boundedness of $H_{\Omega,\alpha}$ and $[b,H_{\Omega,\alpha}]$ on homogeneous Herz spaces and Lebesgue spaces for $b\in BMO(\mathbb{R}^n)$ in [11]. For $\Omega=1$, see for example [9,16].