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Commutators of Bilinear Hardy Operators on Two Weighted Herz Spaces with Variable Exponents

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this paper, we obtain the boundedness of bilinear commutators generated by the bilinear Hardy operator and BMO functions on products of two weighted Herz spaces.

Key Words: Hardy operator, commutator, Muckenhoupt, BMO, variable exponent, Herz space.

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1 Introduction

Denote by $L^1_{\text{loc}}(\mathbb{R}^n)$ the set of all complex-valued locally integrable functions on $\mathbb{R}^n$. The n dimensional Hardy operator was introduced by Faris in [28] as

$$Hf(x) := \frac{1}{\Omega_n |x|^n} \int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n \backslash \{0\} \quad \text{for each} \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

where and what follows Ω_n is the volume of the unit ball in $\mathbb{R}^n$. When $n=1$, the Hardy operator was firstly considered in [10]. In [18], Christ and Grafakos showed that the Hardy operator is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. For the boundedness of H on other functions, we refer the reader to the survey [25] by Shanzhen Lu.

For $m \in \mathbb{N}$, suppose $f_1,\cdots,f_m$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, the m-linear Hardy operator is defined by

$$H(f_1,\cdots,f_m) := \frac{1}{\Omega_{mn}|x|^{mn}} \int_{|(y_1,\cdots,y_m)|<|x|} \prod_{i=1}^m f_i(y_i) dy_1 \cdots dy_m, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

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If $b\in L^1_{\rm loc}(\mathbb{R}^n)$, set

$$\|b\|_*:=\sup_B\frac{1}{|B|}\int_B|b(x)-b_B|dx,$$

where the supremum is taken over all balls in $\mathbb{R}^n$, b_B is the mean of b on B, and what follows $|E|$ is the Lebesgue measure of measurable set E in $\mathbb{R}^n$. A function b is called bounded mean oscillation if $\|b\|_*<\infty$. Denote by BMO($\mathbb{R}^n$) the set of all bounded mean oscillation functions on $\mathbb{R}^n$.

The commutator of m-linear Hardy operator is defined by

$$H_{\vec{b}}(f_1,\cdots,f_m)(x):=\sum_{i=1}^m H^i_{b_i}(f_1,\cdots,f_m)(x),$$

where

$$H^i_{b_i}(f_1,\cdots,f_m)(x):=b_i(x)H(f_1,\cdots,f_m)(x)-H(f_1,\cdots,f_{i-1},f_ib_i,f_{i+1},\cdots,f_m)(x).$$

When $m=1$, the operator $H_b(f)(x)=b(x)Hf(x)-H(bf)(x)$. Although, our results hold for each $m\geq 2$, for brevity, we only consider $m=2$.

Fu, Liu and Lu [31] studied the boundedness of the commutators of weighted Hardy operators (with symbols in BMO($\mathbb{R}^n$)) on $L^p(\mathbb{R}^n)$, where $1<p<\infty$. Fu, Gong and Lu, ect. [34] proved the boundedness of the commutators of weighted multilinear Hardy operators (with symbols in central BMO space) on the product of central Morrey spaces $\dot{B}^{p,\lambda}$. Shi and Lu [27] introduced some characterizations of $\dot{C}^{p,\lambda}$ for $-1/p<\lambda<0$, via the boundedness of commutator operators of Hardy type. Zhao and Lu [7] precisely evaluate the operator norm of the fractional Hardy operator $\mathbb{H}^\beta$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $0<\beta<n$, $1<p<q<\infty$ and $1/p-1/q=\beta/n$. Zhao, Fu and Lu [8] gived some new properties of M_p weight functions on $\mathbb{R}^n$ and used them to characterize the boundedness of bilinear Hardy inequalities on the weighted Lebesgue spaces. Yu and Lu [29] studied the H^1-boundedness of the generalized commutators of Hardy operator with a homogeneous kernel $\mathcal{H}^m_{\Omega,A,\beta}$. Lu, Yan and Zhao [26] explicitly worked out the bounds of the operator $\mathbb{H}$ fromL^p to L^q and from L^1 to $L^{\frac{n}{n-\beta},\infty}$. Fu, Wu and Lu proved the boundedness of commutators generated by the p-adic Hardy operators (Hardy-Littlewood-Pólya operators) and the central BMO functions on $L^q(|x|^\alpha_p dx)$ and Herz spaces $K^{\alpha,q}_r$. Zhao, Fu and Lu [9] proved that the higher dimensional Hardy operator is bounded from Hardy spaces to Lebesgue spaces, and discussed the endpoint estimate for the commutators generated by the Hardy operator and (central) BMO functions. Fu, Grafakos and Lu, etc. [32] obtained norms of m-linear Hardy operators and m-linear Hilbert operators on Lebesgue spaces with power weights.

In [19], Izuki proved the boundedness of commutators generated by singular integrals and BMO functions on Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$. Shu, Wang and Meng [17] obtained the boundedness of commutators of Hardy type operators