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Borderline Weighted Estimates for Commutators of Fractional Integrals

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. Let $I_{\alpha,\vec{b}}$ be the multilinear commutators of the fractional integrals I_α with the symbol $\vec{b} = (b_1, \cdots, b_k)$. We show that the constant of borderline weighted estimates for I_α is $\frac{1}{\varepsilon}$, and for $I_{\alpha,\vec{b}}$ is $\frac{1}{\varepsilon^{k+1}}$ with each b_i belongs to the Orlicz space $Osc_{\exp L^{s_i}}$.

Key Words: Commutators, fractional integrals, borderline weighted estimates, Fefferman-Stein inequality.

AMS Subject Classifications: 42B25, 47G10

1 Introduction

Let M be the Hardy-Littlewood maximal function, which is defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q containing x in $\mathbb{R}^n$ with the sides parallel to the coordinate axes. Since the 1930s, there have been many outstanding works in the study of the Hardy-Littlewood maximal function. Among such achievements are the celebrated works of Hardy, Littlewood and Wiener, Fefferman and Stein [9], and Muckenhoupt [13]. Recall that the Hardy-Littlewood-Wiener theorem states that M is bounded

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from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ $(1 < p \leq \infty)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, and the Fefferman-Stein inequality [9] can be expressed in the way that

$$\|Mf\|_{L^{1,\infty}(\omega)} \leq C \int_{\mathbb{R}^n} |f| M\omega dx. \tag{1.1}$$

The question whether one can extend inequality (1.1) to other type of operators, such as the Hilbert transforms and the Calderón-Zygmund singular intergrals, is known as Muckenhoupt and Wheeden conjecture. In 2012, Reguera and Thiele [22] surprisingly showed that the Muckenhoupt and Wheeden conjecture was not true for the Hilbert transform, which fully indicates that the Hilbert transform does not enjoy the similar weak type inequality as in (1.1). In 1994, Pérez [17] obtained the following less fine inequaltiy for the Calderón-Zygmund singular intergrals.

$$\|Tf\|_{L^{1,\infty}(\omega)} \leq C_{\epsilon,T} \int_{\mathbb{R}^n} |f| M_{L(\log L)^\epsilon} \omega dx, \quad \omega \geq 0, \quad \epsilon > 0. \tag{1.2}$$

Since then, efforts have been made to clarify and separate the constant $C_{\epsilon,T}$. It was Hytönen and Pérez [10] who first showed that the constant can be gained is ε^{-1} for T and its corresponding maximal singular integral operators T^*. Recently, Domingo-Salazar, Lacey, Rey [8] generalized the results in [10] and further proved that T^* is bounded as a map from $L^1(M_{L \log\log L (\log\log\log L)^\alpha} w)$ into weak-$L^1(w)$ for $1 < \alpha < 2$ and the constant can be obtained is $(\alpha - 1)^{-1}$.

Now we turn to the background of the commutators of T, which can be traced back to the celebrated works of Coifman, Rochberg and Weiss [3]. For a suitable smooth function f, the commutator of T is defined as $[b, T]f = T(bf) - bT(f)$. In [3], the authors proved that if b belongs to BMO($\mathbb{R}^n$), then [b,T] is bounded from $L^p(\mathbb{R}^n)$ onto itself $(1 < p < \infty)$. Conversely, if all commutators of Riesz transform $[R_j, b]$, $1 \leq j \leq n$, are L^p bounded, then $b \in$ BMO($\mathbb{R}^n$).

In 1995, Pérez [18] pointed out that the commutators of CZOs are not weak type $(1, 1)$ operators. As a replacement, he gave the following $L \log L$ endpoint estimate:

$$\omega\big(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}\big) \leq C_{\|b\|_{BMO}} [w]_{A_1} \int_{\mathbb{R}^n} \Phi\big(|f|/\lambda\big) \omega dx, \tag{1.3}$$

where $\Phi(t) = t(1 + \log^+ t)$, $\omega \in A_1$.

Quite naturally, one may ask whether the commutators $[b, T]$ still enjoy the similar inequality as in (1.2) or not. In 2001, Pérez and Pradolin [19] established the following inequality for arbitrary non negative weights w.

$$\omega\big(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}\big) \leq C_{T,\varepsilon} \int_{\mathbb{R}^n} \Phi\big(|f| \|b\|_{BMO}/\lambda\big) M_{L(\log L)^{1+\varepsilon}} \omega dx. \tag{1.4}$$

In 2017, Pérez et al [20] further figured out that the constant in (1.4) is $\frac{C_T}{\varepsilon^2}$, that is

$$\omega\big(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}\big) \leq \frac{C_T}{\varepsilon^2} \int_{\mathbb{R}^n} \Phi\big(|f| \|b\|_{BMO}/\lambda\big) M_{L(\log L)^{1+\varepsilon}} \omega dx. \tag{1.5}$$