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Intermittent Behaviors in Coupled Piecewise Expanding Map Lattices

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Abstract. In this paper, we propose a new method to study intermittent behaviors of coupled piecewise-expanding map lattices. We show that the successive transition between ordered and disordered phases occurs for almost every orbit when the coupling is small. That is,

$$\liminf_{n\to\infty} \sum_{1\le i,j\le m} |x_i(n)-x_j(n)| = 0,$$
$$\limsup_{n\to\infty} \sum_{1\le i,j\le m} |x_i(n)-x_j(n)| \ge c_0 > 0,$$

where $x_i(n)$ correspond to the coordinates of m nodes at the iterative step n. Moreover, when the uncoupled system is generated by the tent map and the lattice consists of two nodes, we prove a phase transition occurs between synchronization and intermittent behaviors. That is,

$$\lim_{n\to\infty} |x_1(n)-x_2(n)| = 0 \quad \text{for} \quad \left|c-\frac{1}{2}\right| < \frac{1}{4}$$

and intermittent behaviors occur for $|c-\frac{1}{2}| > \frac{1}{4}$, where $0 \le c \le 1$ is the coupling.

Key Words: Synchronization, pseudo synchronization, phase transition, Coupled map Lattices, piecewise expanding map.

AMS Subject Classifications: 37D20, 37A25

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1 Introduction

In this paper, we study the intermittent dynamical behavior of coupled piecewise-expanding map lattices. Let $f : [0,1] \to [0,1]$ be a piecewise expanding map, I be the $m \times m$ identity matrix and A be an $m \times m$ symmetric matrix satisfying $A\mathbf{e} = 0$, where $\mathbf{e} = [1, \cdots, 1]^\top$. Consider the dynamical system defined by a coupled map lattice:

$$T : \quad \mathbf{x}(n+1) = (I + cA)\mathbf{f}(\mathbf{x}(n)), \tag{1.1}$$

where c is the coupling coefficient, $\mathbf{x}(n) = [x_1(n), \cdots, x_m(n)]^\top \in [0,1]^m$ for $n \in \mathbb{N} \cup \{0\}$ and $\mathbf{f}(\mathbf{x}(n)) = [f(x_1(n)), \cdots, f(x_m(n))]^\top$. In case of no confusion, we also use bold letters $\mathbf{x}$ or $\mathbf{p} = (x_1, \cdots, x_m)$ to denote points in $[0,1]^m$.

Because of $A\mathbf{e} = 0$, it can be easily seen that the diagonal $D_{\rm syn} = \{(x_1, \cdots, x_m) \in [0,1]^m \mid x_1 = \cdots = x_m\}$ is an invariant set for synchronized points of T. An interesting question on the dynamical behavior of the coupled map lattice (1.1) can be raised as whether $D_{\rm syn}$ is a global attractor, or equivalently, whether synchronization occurs for (1.1). There have been plenty of results on the study of synchronization when f generates a chaotic dynamical system. Common examples include the tent maps and the Logistic maps, one can see [2, 20, 27] and references therein. It has been shown in these results that chaotic synchronization can occur only if c is far from zero. That is, chaotic synchronization can not occur for small coupling strength.

However, a more complicated phenomenon has been found by numerical simulations when c is out of the synchronized region. Roughly speaking, it is found that a typical orbit can enter into and exits slowly from an arbitrarily small neighborhood of $D_{\rm syn}$ for infinite times. In other word, the successive transition between being close to the diagonal and being far from the diagonal can happen. We call this phenomenon as pseudo-synchronization.

The pseudo-synchronization is closely related to the clustering phenomenon in global coupled map lattices by Kaneko et al. [4, 9–14]. In numerical experiments, it showed that when (1.1) is a globally coupling system with large m, elements differentiate into some clusters, and elements in each cluster oscillate synchronously, while the behaviors in different clusters are various. Moreover, the differentiation by clustering is a temporal behavior in nature [14]. One can easily see that the pseudo-synchronization is a special case of the temporal clustering. In fact, the temporal clustering is also found in all systems of (1.1) with small c. Similar behaviors were also widely explored in weakly coupled continuous-time chaotic systems. For example, the successive transition between bursting and spiking was discovered in the study of epilepsy, see [5, 6] and references therein. More related results are shown in [3, 7, 24, 25] and references therein. To provide a mathematical proof for the mechanism of pseudo-synchronization for coupled map lattices is one of motivations of this paper.

On the other hand, there are a series of mathematical results on dynamical behaviors of weakly-coupled map lattices. In [15], Keller showed that the existence of unique